How To Pump A Swing

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Abstract

The analysis of pumping a swing has been done in the standing mode and seated mode in different ways using Mathematica. It's shown that the standing mode is equivalent to a parametric oscillator while the seated mode is equivalent in a sense to a driven oscillator. The description of both modes from a qualitative point of view is given. The other main differences between both modes are illustrated during the analysis.

Contents

1-Introduction

A swing is a famous and an interesting tool that children like to play with. A swing consists mainly of a long rod ended with a seat on which the child can stand or sit while swinging. Normally friction is a major dissipative force that affects the motion of a swing causing it to decay gradually. So a child playing with a swing has to learn some pumping mechanism in order to keep the swing running. Not only can a child keep the swing running but he can also make the oscillation grow up. Surprising enough, those pumping schemes can even start a swing from the rest positions as experienced by children on the playground. Pumping means that the child keeps storing energy into the swing by performing some movements at certain positions. This energy by means of conservation of momentum is converted into kinetic energy. The mechanism of pumping may take several forms depending on the state of the child (standing on the swing or sitting.) A child can pump from a standing position by periodically standing and squatting on the swing which results in periodic displacement of the center of mass up and down. This is modeled by a changing the length of the rod of the swing with time. Another method is by leaning forward and backward periodically while sitting on the swing.Both schemes will be analysed in this paper.

2- Pumping a swing from a standing position

As mentioned in the introduction, the swing in this scheme is modeled as a pendulum having its length as a function of time. The child is modeled as a point mass at the end of the rod moving up and down. This motion is shown in figure (1).

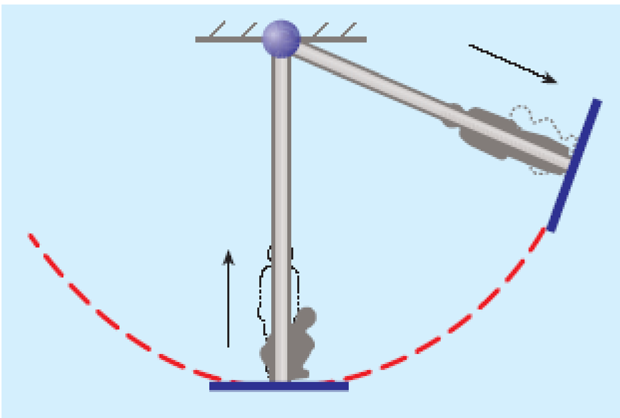


Figure (1)

Neglecting the mass of the rod, the kinetic energy of the swing is $T = \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}m\dot{r}^2$ and the potential energy is: $V = mgr(1-\cos(\theta))$ and the Lagrangian becomes $L = \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}m\dot{r}^2 + mgr\cos(\theta)$ after neglecting the constant term in the potential. There is a force of constraint represented by the tension in the rod in the r direction which forces the length of the swing to be not greater than the length of the rod.

This will introduce an unkown force of constraint in the equation of motion of **r**. The equation of motion for the angle θ – measured from the midpoint- is :

$r\ddot{\theta}+2\dot{r}\dot{\theta}=-g\sin\theta$. As an initial investigation of this equation I tried to change the length periodically as a sin wave: $l=l_0+\alpha\sin(\omega t)$ and scanned against the frequency ω, I found that the amplitude of the oscillation was actually amplified in a certain range of frequencies, but maximum amplification occurred when ω was 1.9 times the natural frequency of the swing $\sqrt{\frac{g}{l}}$ as shown figure (1). The length of the swing was chosen to be 2 m (that makes the natural frequency of the swing ω_0 =2.2 rad/s) and α to be 0.25). Setting $\dot{\theta}(0)$ to be 0 to emulate the rest condition at $\theta(0)$=0, I found that an amplification in the angle began to occur at pumping frequency of 4.2 rad/s (approximately double the natural frequency of the swing.) We notice that the oscillation is becoming unstable as time increases since this is not the realistic case (the child doesn't move with a constant frequency but adjusts his motion according to the motion of the swing) and because I excluded the drag of the air and friction of the swing from this analysis. This proves that a swing can be pumped from very small initial perturbation by a continuous motion of the child in principle. This is called a parametric oscillator, where energy is pumped into the oscillator not by a force varying with time but by a variation in one of the parameters of the equation.

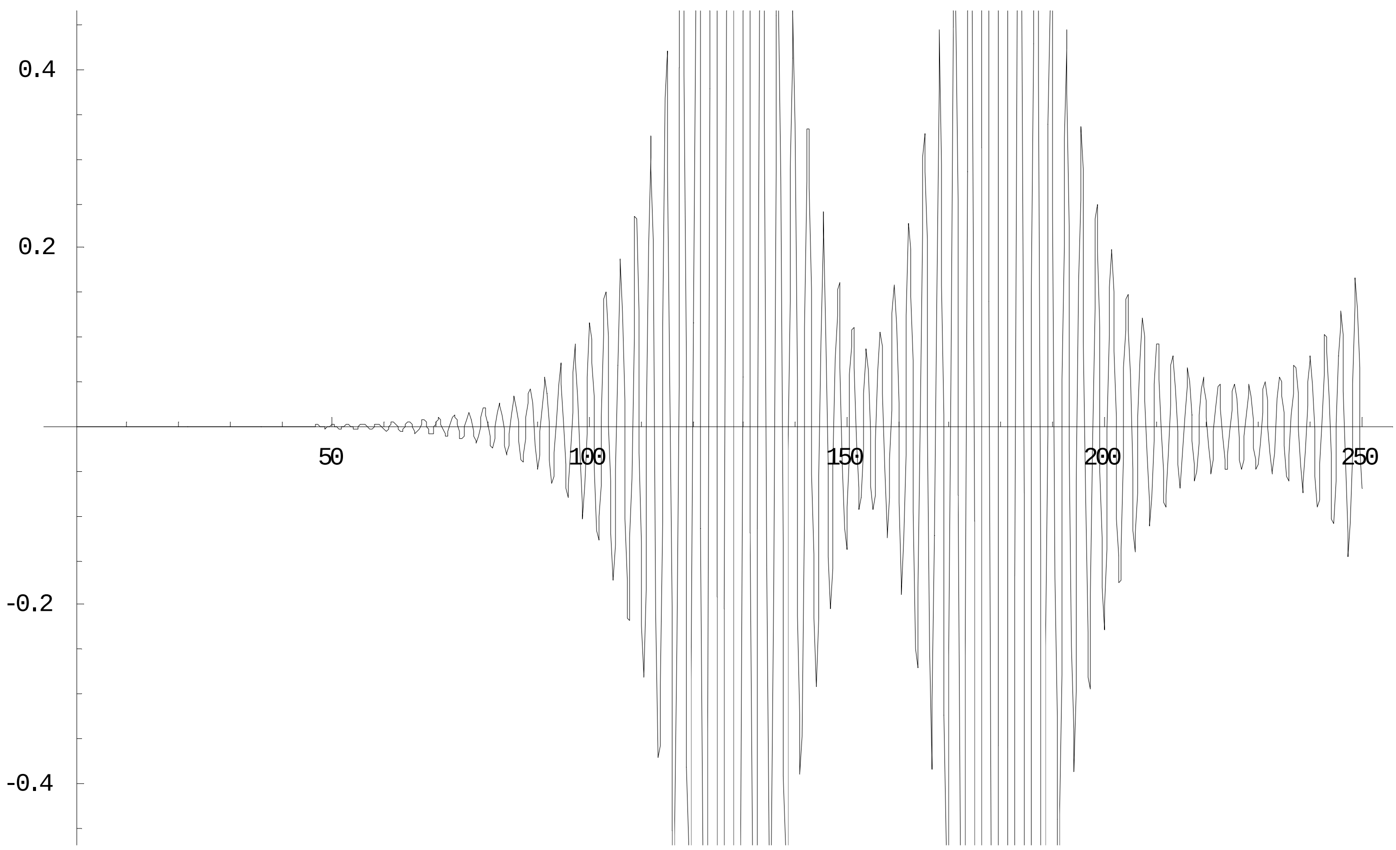


Figure (2)

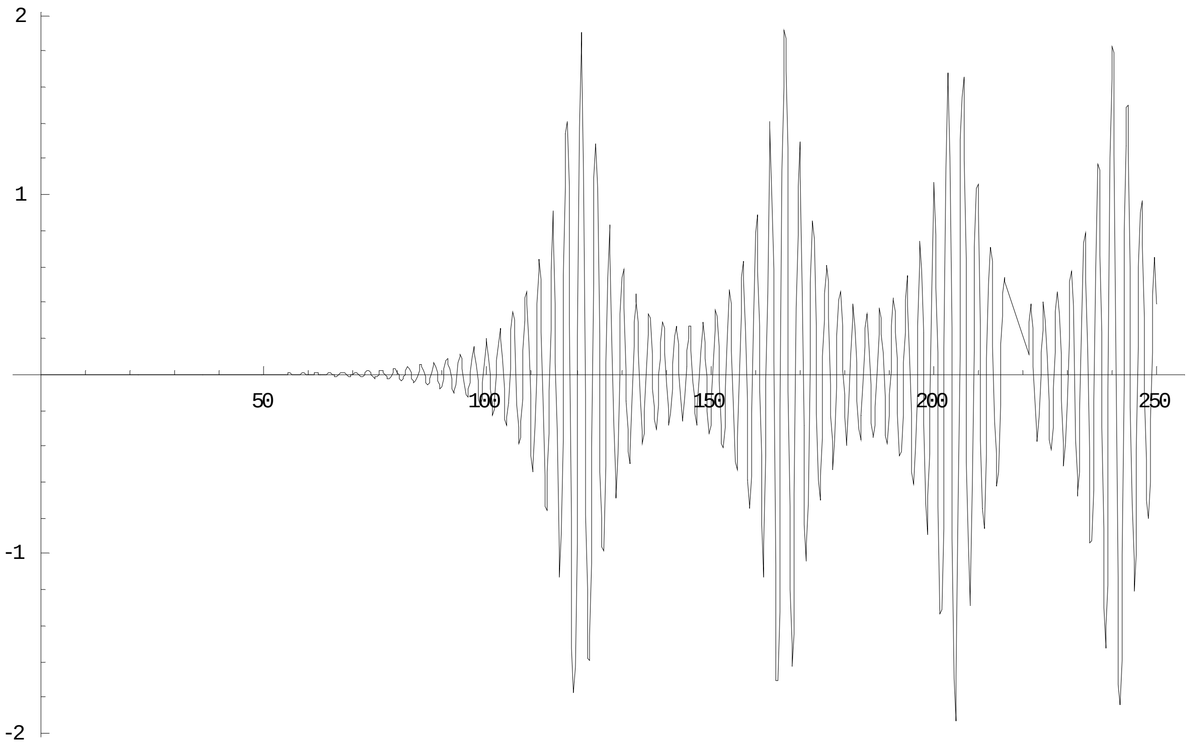


Figure (3)

I noticed by scanning the pumping frequency above this value(4.25 rad/sec) that the response of the swing becomes sensitive to the pumping frequency and periods of amplification and attenuation occur at frequencies that depend very sensitively on the pumping frequency. The response in figures (2), (3) corresponds to pumping frequencies that are .01 rad/s different from each other.

The attenuation may be interpreted to be due to the gradual change of the frequency of the swing until a big phase shift is built between it and the pumping signal, we emphasize again that this model has a lot of approximation. A closer look at the response of the swing illustrates this fact.

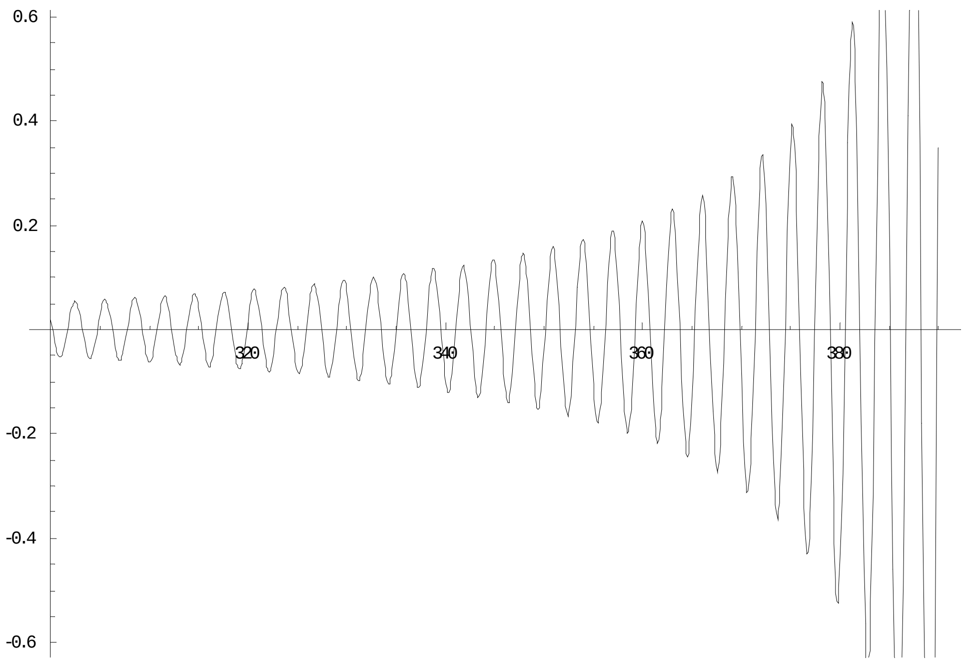


Figure (4)

I tried to use a more realistic model where the child doesn't move with a constant frequency but changes his motion according to the position of the swing, i.e $l = r = l_0 - \alpha Cos(\theta)$. The variation of the length in this case looks like:

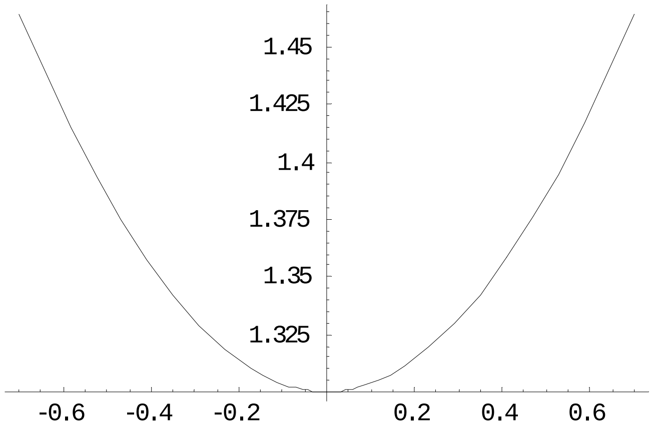


Figure (5)

In this case the equation of motion for the θ angle becomes $[l_0 - \alpha Cos(\theta)]\ddot{\theta} + 2\alpha Sin(\theta)\dot{\theta}^2 = -g\sin\theta$. As shown in figure (6), numerical solution of the equation of motion showed no amplification in θ. This is not surprising since the energy pumped by the child in one quarter cycle is restored in the next quarter cycle.

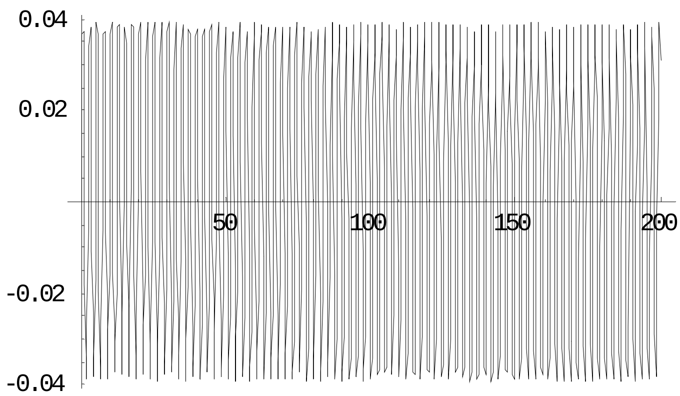


Figure (6)

Now, coming to the most realistic model of the motion of the child in standing position that gave me the most practical results: I defined a new exponential function that models the variation of r as a function of θ. The function is a multi-valued function since the length of the swing depends not only on the value of θ but on the direction of the change of θ with time as shown in figure (7). I can control the steepness of the change of the length around the mid-point by controlling the exponential growth rate.

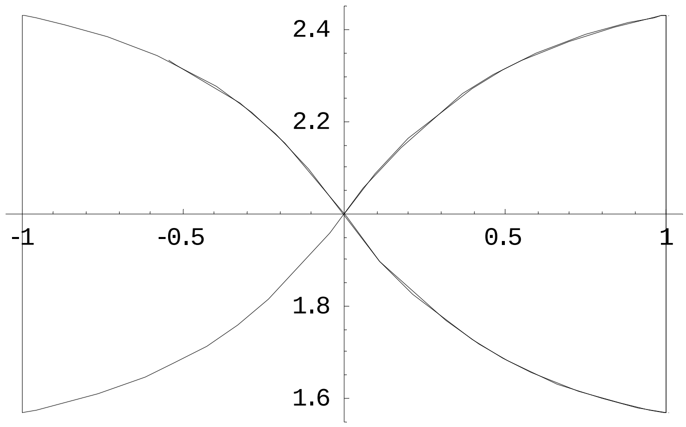


Figure (7)

An example of the amplification generated by this modulated length is shown in figure (8).

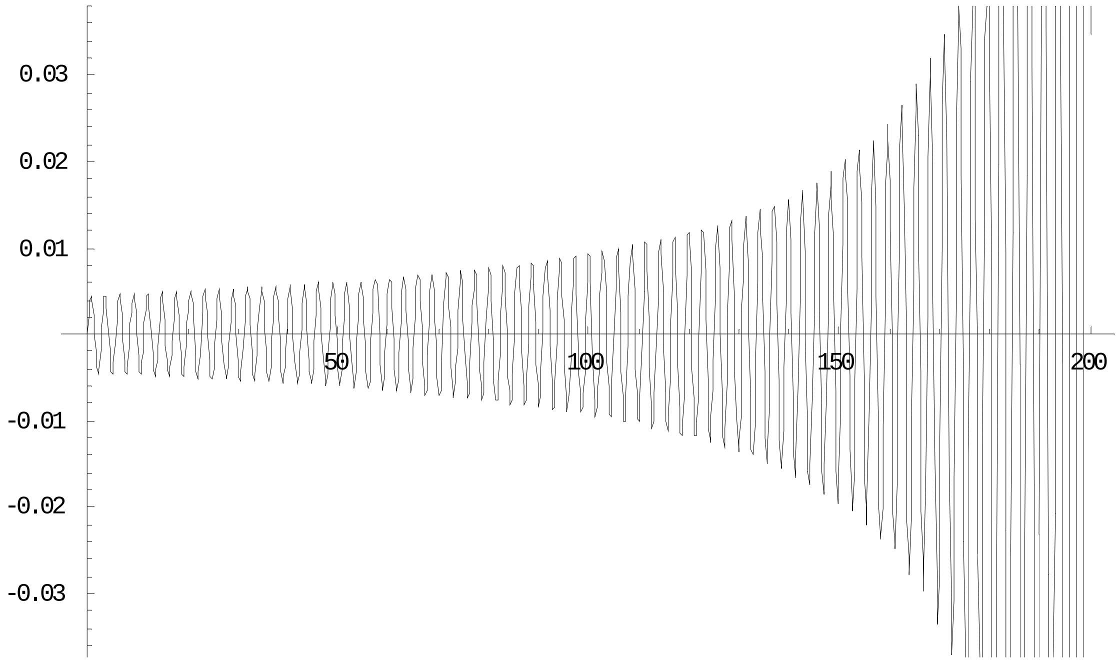


Figure (8)

An interesting phenomenon was observed, which is that the smaller the initial angular velocity of the swing, the larger the threshold steepness of the variation of length needed to start the amplification process. The previous figure, is for initial angular velocity = 0.01 rad/sec. For the same length variation function (same steepness) and initial angular velocity = .001 rad/sec we find the response given in figure (9).

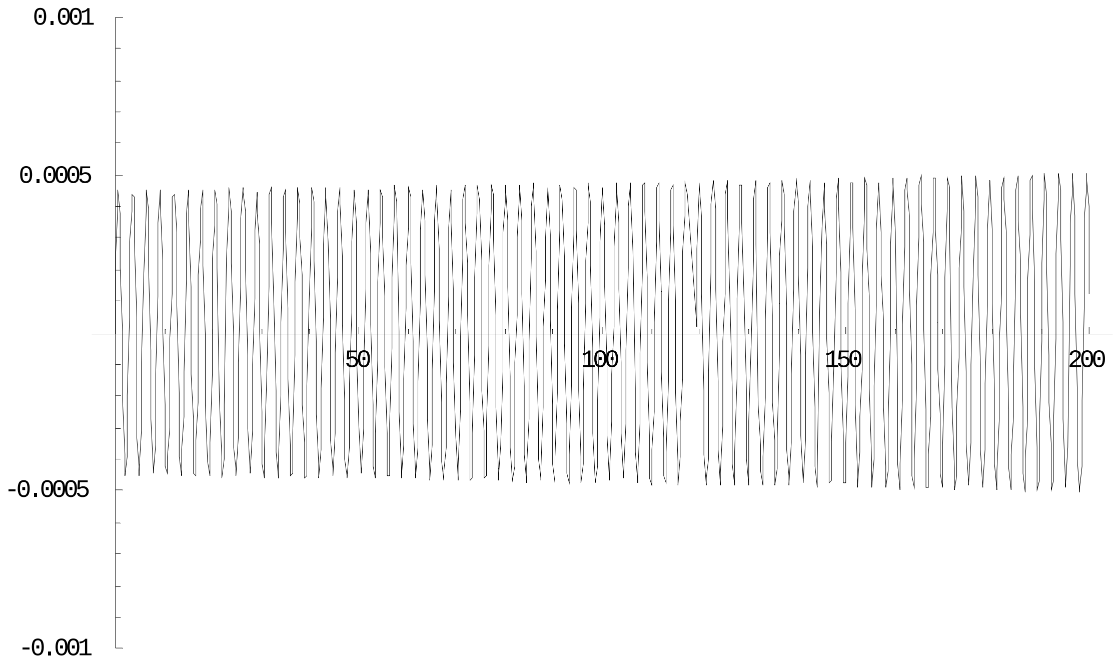


Figure (9)

For more detail on this point see Mathematica code in the Appendix.

Now let's try to figure out qualitatively how the movement of the child at the midpoint of the swing increases the height of the swing. As before, I will assume that the center of gravity (the child) is moving up and down but I'll consider the extreme case where the child stands up instantaneously at the mid-point and squats instantaneously at the highest point. We can look at this problem from two perspectives:

2.1 The conservation of angular momentum

As shown in figure (10) the motion of the child causes an immediate change of the radius of oscillation at the point where the speed is maximum.

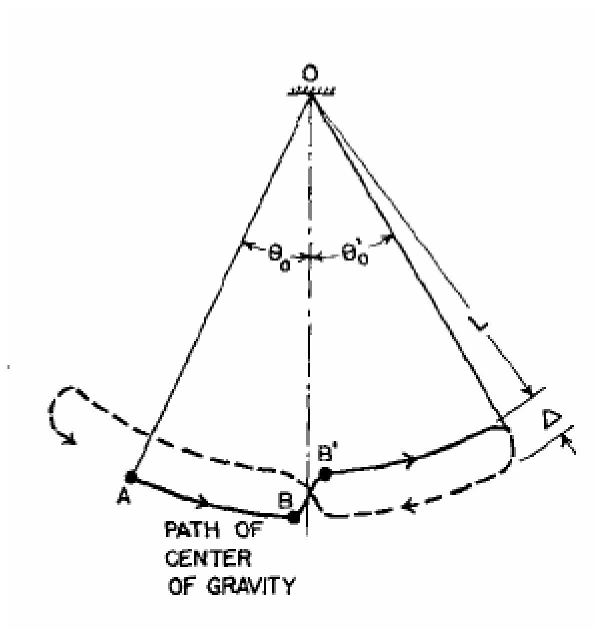


Figure (10)

Conservation of angular momentum states that the angular speed should increase from the point B to B' such that the angular momentum stays the same. So from the conservation of angular momentum we get $m r_{B'}^2 \dot{\theta}_{B'} = m r_B^2 \dot{\theta}_B$ hence $\frac{\dot{\theta}_B}{\dot{\theta}_{B'}} = (1-\frac{\Delta}{r_B})^2$ where Δ is the difference between r_B, $r_{B'}$. From the conservation of energy, we can get $\theta_0'(n)$, which is the maximum displacement in θ as a function of $\dot{\theta}_{B'}$ as: $\frac{1}{2} m r_{B'}^2 \dot{\theta}_{B'}^2(n) = mgr_{B'}(1-\cos\theta_0'(n))$

Hence the relation between the maximum angle θ_0' in two consecutive cycles n, n+1

is $\frac{(1-\cos\theta_0'(n+1))}{(1-\cos\theta_0'(n))} = \frac{\dot{\theta}_{B'}^2(n+1)}{\dot{\theta}_{B'}^2(n)}$.

So beginning from an initial displacement angle θ_0 we can get the increase in the θ_0 as a function of the cycle number as follows:

Since in one complete cycle, the maximum angular speed $\dot{\theta}_{B'}$ increases by a factor: $(1-\frac{\Delta}{r_B})^{-4}$ then $\frac{(1-\cos\theta_0'(n+1))}{(1-\cos\theta_0'(n))} = (1-\frac{\Delta}{r_B})^{-8}$. Solving recursively for θ_0' through this equation beginning from θ_0'=0.00001 (approximate rest condition) I found that it grew very rapidly and that θ_0' reached π only after 15 cycles !! Of course this is not a realistic model, and hence the nonrealistic results are not surprising.

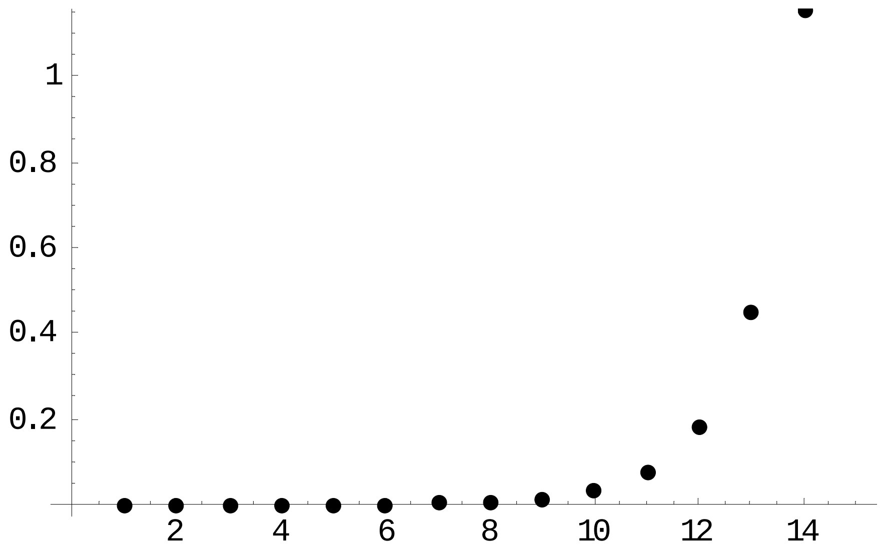


Figure (11)

2.2 Conservation of energy

By considering the extreme case where the child moves instanteously up and down in the mid-point and extreme points respectively we can calculate how much energy is pumped into the swing per cycle. Let the displacement of the child be ∇x. When the child stands up in the mid point he is pumping potential energy into the swing by doing work opposing to two forces :

a- the gravitational force $\nabla W = mg.\nabla x$

b- The centrifugal force $\nabla W = \frac{mv^2}{l}\nabla x$

Where v is the maximum velocity of the swing, and l can be taken to be the average length.

While when the child sits down at the extreme point he is giving part of his potential energy in favor of the gravitational force only, the centrifugal force is zero since the velocity is zero. Hence $\nabla W = -mg.\nabla x$. Therefore we see that the energy stored in the swing in one cycle equals $2\frac{m{v_{\max}}^2}{l}\nabla x$

This will lead to the increase of the maximum velocity each cycle, and hence an exponential increase in the stored energy in the swing. A simple algebra shows that the increase in the maximum velocity in half cycle, $v^2(n+\frac{1}{2}) = v^2(n)(1+2\frac{\nabla x}{l})$. Hence in one complete cycle, the velocity increases by a factor $(1+2\frac{\nabla x}{l})$. After n cycles, we find the velocity given in terms of the initial velocity is $v(n) = v(0)(1+2\frac{\nabla x}{l})^n$

By expressing the kinetic energy at the mid-point with the difference in potential energy expressed in terms of the maximum swinging angle we get: $\frac{1}{2}mv^2[n] = mgl(1-\cos(\theta_0(n))$ and hence the maximum angle θ is given after n cycles in terms of the initial maximum displacement θ[0] as : $\frac{(1-\cos(\theta_0(n))}{(1-\cos(\theta_0(0))} = (1+2\frac{\nabla x}{l})^{2n}$ or equivalently $\frac{(1-\cos\theta_0'(n+1))}{(1-\cos\theta_0'(n))} = (1+2\frac{\nabla x}{l})^2$. We notice that this result is different from the previous case (from angular momentum considerations) to a first order in $\frac{\nabla x}{l}$ by a factor 2 in the exponent !! This is one conflicting result in this paper. Using the same initial angle and displacement as in the previous case we see that the maximum angle increase with time as shown in figure (12)

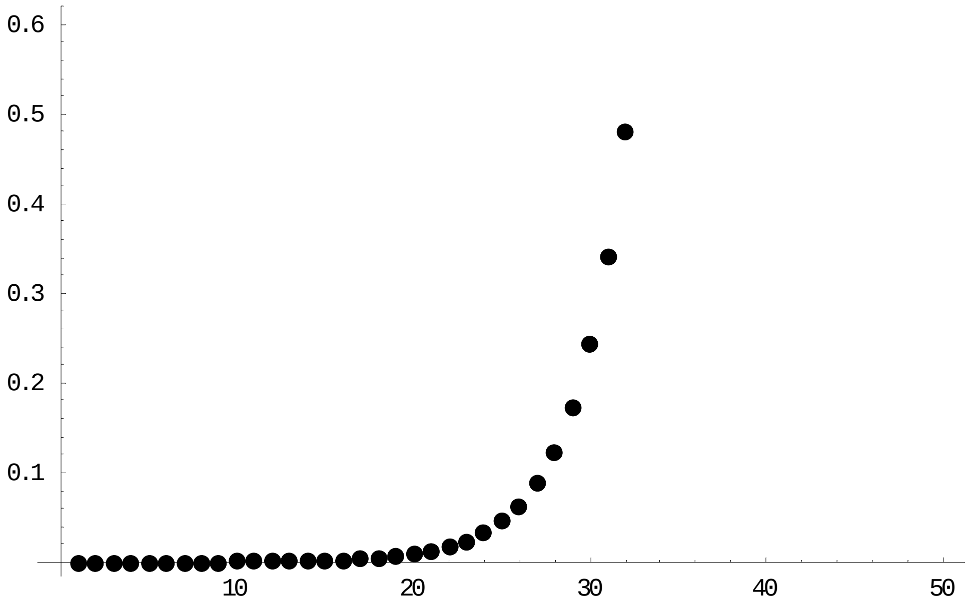
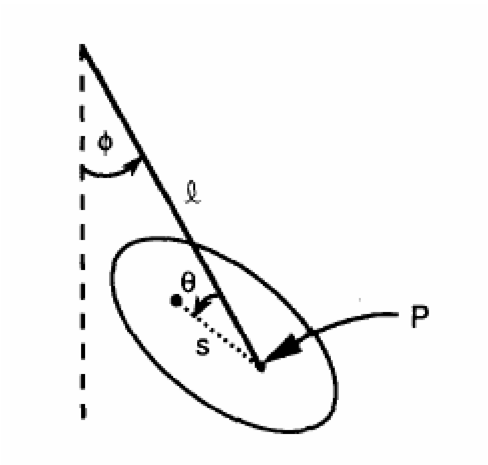


Figure (12)

We note in this discussion that the frame of the swing has no role in the pumping process, since the tension exerted from the hanging point doesn't appear in the equations and it's always normal to the velocity of the center of mass. However, it plays an important rule in fixing the swing to the ground, since without it the whole swing will be swinging in the opposite direction to the motion of the child to conserve the angular momentum.

2.3 Another mode for pumping from a seated position

Another mode of pumping the swing from a standing position is by leaning backward and forward while standing as shown in figure (13).



Figure(13)

The swinger is modeled by a single mass at the center of mass rotating around the lowest point of the swing independently. The angle between the swing and the swinger is called θ (I took it in the other direction). The potential energy is given by $V = mg[-LCos(\phi) + lCos(\theta - \phi)]$ and the kinetic energy by $T = L^2\dot{\phi}^2 + l^2(\dot{\theta} + \dot{\phi})^2 + 2Ll\dot{\phi}(\dot{\theta} - \dot{\phi})$

Hence the equation of motion in the φ direction is

$$mgLSin(\phi)+mglSin(\theta-\phi)+[2L\ddot{\phi}-2l^2(\ddot{\theta}-\ddot{\phi})+2LlCos\theta(\ddot{\theta}-2\ddot{\phi})-2Ll\dot{\theta}(\dot{\theta}-2\dot{\phi})Sin\theta=0 .$$

Simulating the motion of the swinger by a sinusoidal change using mathematica showed that this mode is capable to start the swing from rest but the oscillations are of constant amplitude and are not amplified exponentially like the previous case.

3 Pumping a swing from a seated position

Another scheme of pumping a swing, is by pumping from the seated position. This involves a sudden rotation of the rider's body when the swing momentarily comes to either of the two stops. A model for the rider and the swing in this scheme is shown in figure (14).

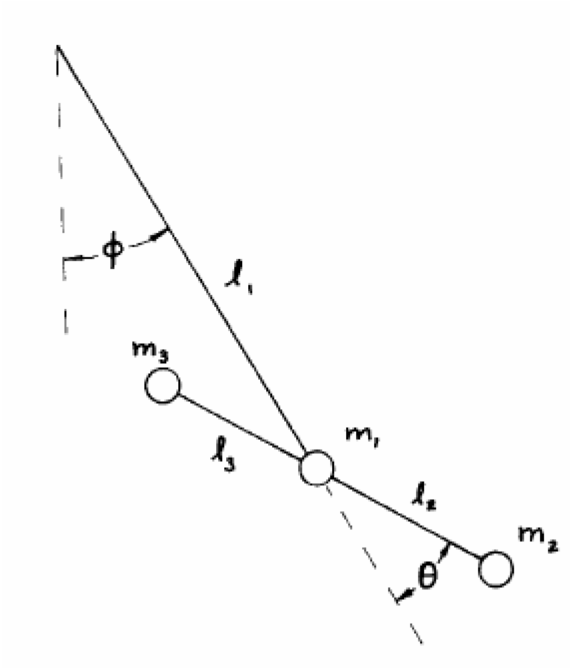


Figure (14)

Similar to the double pendulum , with an additional mass, the potential energy of the three masses (P.E)

$$=-m_1l_1Cos\phi-m_2(l_2Cos(\theta+\phi)+l_1Cos\phi)-m_3(l_1Cos\phi-l_3Cos(\theta+\phi))$$

And their kinetic energy (K.E) =

$$\frac{1}{2}m_1l_1^2\dot{\phi}^2+\frac{1}{2}m_2[l_1^2\dot{\phi}^2+l_2^2(\dot{\phi}+\dot{\theta})^2+2l_1l_2\dot{\phi}(\dot{\theta}+\dot{\phi})Cos(\theta)]+\frac{1}{2}m_3[l_1^2\dot{\phi}^2+l_3^2(\dot{\phi}+\dot{\theta})^2-2l_1l_3\dot{\phi}(\dot{\phi}+\dot{\theta})Cos(\theta)]$$

Hence the Lagrangian can be written as

$$L=\tfrac{1}{2}I_1\dot{\phi}^2+\tfrac{1}{2}I_2(\dot{\phi}+\dot{\theta})^2-l_1N\dot{\phi}(\dot{\phi}+\dot{\theta})\cos\theta+Ml_1g\cos\phi-Ng\cos(\phi+\theta),$$

where

$$M=m_1+m_2+m_3,$$
$$I_1=Ml_1^2,$$
$$I_2=m_2l_2^2+m_3l_3^2,$$
$$N=m_3l_3-m_2l_2.$$

The equation of motion of the ϕ angle is

$-M I_1 g Sin(\phi(t))+N g Sin(\phi(t)+\Theta(t))-I_1 \phi''(t)-I_2(\phi''(t)+\Theta''(t))-2 I_1 N \phi''(t) Cos(\Theta(t))-I1 N \Theta''(t)Cos(\Theta(t))==0$

When solving this equation numerically using Mathematica, I got a surprise. Beginning from a very small angular velocity', ϕ' (t)=.001 rad/sec, the angular displacement has already grown up. But unlike the standing position pumping scheme, the growth is a linear growth not an exponential growth.

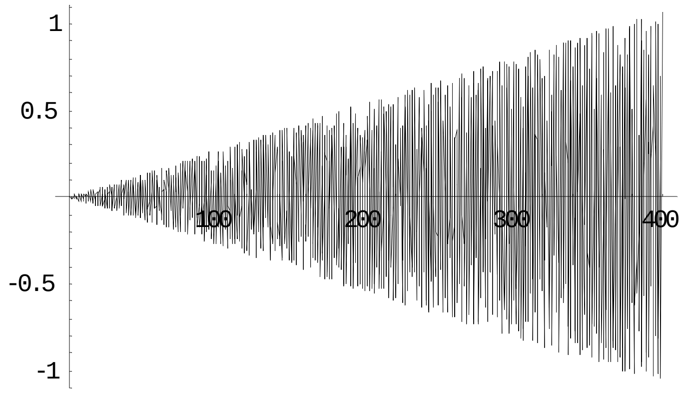


Figure (15)

The Θ(t) was taken in this simulation to be a constant value of .5 rad while ϕ(t) was increasing (the swing moving to the right) and -0.5 rad/sec while ϕ(t) increasing. When I changed the values of Θ(t) to be 0.7 rad/sec, the oscillation has grown faster as expected. This is shown in figure (16)

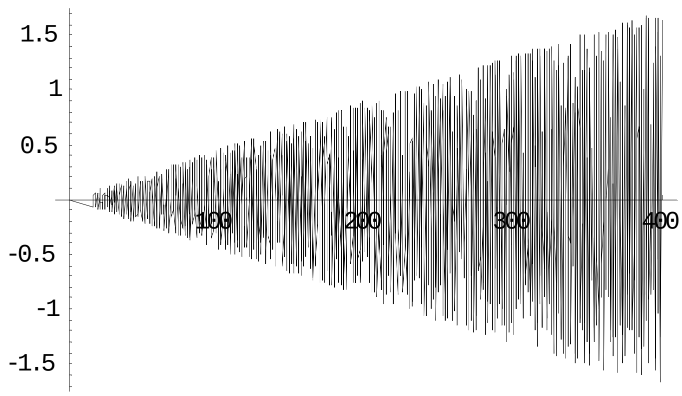


Figure (16)

A special case occurs when $m_2 I_2 - m_3 I_3 = 0$ then we find that N=0 and the Lagrangian reduces to $\frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 (\dot{\phi} + \dot{\theta})^2 + M l_1 g Cos(\phi)$. The equation of motion then becomes :

$(I_1 + I_2)\ddot{\phi} + M g l_1 Sin(\phi) = -I_2 \ddot{\theta}$. It's clear that the pumping of the swing by changing θ periodically is equivalent to a driven harmonic oscillator unlike the pumping from a standing position which is equivalent to a parametric oscillator.

By modeling the variation of θ by a fixed frequency, I found that the oscillation was amplified (linearly) and succession of amplification and attenuation occurred due to the phase difference between the driving θ and the oscillating φ as in the case of parametric amplification.

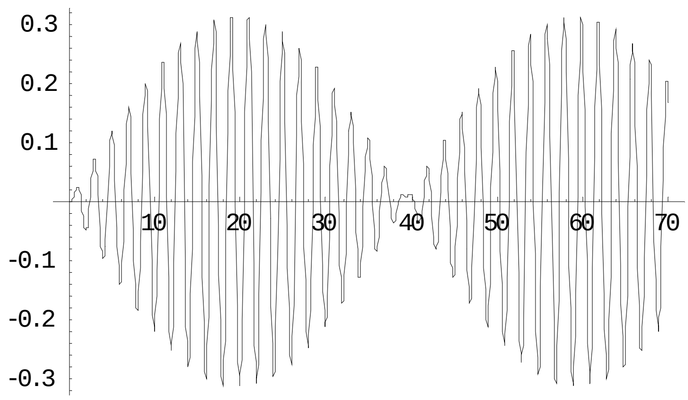


Figure (17)

Unlike the standing position pumping, the maximum amplification occurred near the natural frequency of the swing (not double the natural frequency) approximately at $\sqrt{g/l} - .13$.

As we did with the standing position, trying to figure out qualitatively what the physical cause of pumping is, we find that the source of pumping energy is in the sudden change of the angular momentum at the highest points. Considering the idealized model, we notice that the change of the orientation of the body of the rider at the highest points represents an increase in the angular momentum in the direction that makes the swing go back. The sudden stop of the motion of the rider forces the swing to go back faster to conserve the total angular momentum. That wouldn't happen of course if the swing wasn't fixed to the ground.

Now let's search for the optimal point of leaning forward and backward. Up to now, we considered that the bending of the child's body takes place at the highest points only. When I considered the other extreme case in which the child bends his body at the lowest point where the velocity if the swing is maximum, I found for my surprise that no amplification happened at all!!

I tried also to to make the $\dot{\theta} = \dot{\phi}$ but found no amplification also!!

One advantage of this pumping scheme (from seated position) is that it's more efficient in starting the swing from rest position. Leaning forward and backward, will cause the swing to move in the other direction to conserve angular momentum even it was absolutely at rest. This is not the case in pumping by standing and squatting.

An interesting observation in this mode is that I noticed, experimentally, is that the oscillation of the swing increases when the swinger stretches his leg in the forward motion and bends them in the backward motion. This supports the conclusion that the increase of the θ increases the growth rate of the oscillations.

To compare the pumping efficiency of both techniques from energy point of view, it's obvious that when excluding dissipation both schemes will have the same power efficiency. Another point of comparison is the growth rate. This means which pumping scheme allows the child to pump a larger amount of energy per cycle? By comparison of figures (2), (16), for the case of a periodic change of the length and the leaning angle, it's obvious that the seated position grows much faster the standing position. But from my experiment on the swing, I felt more tired when I pumped the swing from the standing position than the case of seated pumping!! This is considered another conflicting result.

4- Conclusion

The two different pumping schemes has been illustrated and analyzed in this paper. While the standing position pumping is equivalent to a parametric oscillator, the seated position is equivalent to a driven oscillator. Although the oscillation grows up exponentially for the standing position and linearly for the seated position, it was shown that the pumping from seated position is more effective in terms of oscillation growth rate.

The actual pumping mechanisms followed on the playground, I guess, are combinations of both techniques since children , specially experienced ones, lean forward and backward while standing and squatting. This may imply that the best pumping scheme is neither of the two, but a combination of them.

5- Acknowledgment

I thank my friend Abdu Alaswad, who helped me try the pumping on a swing and let me have the sense of pumping from the seated and the standing position. He took a ride as well and gave me the idea of the other mode of pumping from the standing position that I have included in page (9).

6- References

1. The pumping of a swing from the seated position, William B. Case and Mark A. Sawnson Am. J. Phys, Volume 58, Issue 5, pp. 463-467
2. The pumping of a swing from the standing position, William B. Case, Am. J. Phys, Volume 64, Issue 3, pp. 215-220
3. Pumping on a swing, Peter L. Tea and Harold Flak, Am. J. Phys, Volume 36, Issue 12, pp. 1165-1166
4. How children swing, Stephen M. Curry, Volume 44, Issue 10, pp. 924-926

Appendix I List of conflicting results

1- The difference between the growth rate of the standing position pumping when analysed from a conservation of energy and conservation of angular momentum points of view.
2- The analysis showed that the amount of energy pumped per cycle by the seated pumping is higher than that of the standing position pumping. My experience says the inverse.
3-