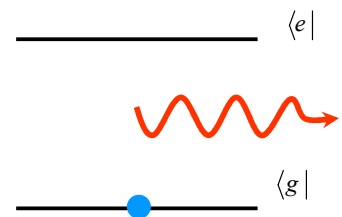
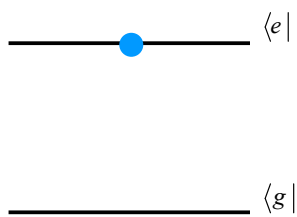


Interaction Between Light and Matter



Prepared by
Tareq Ahmed Mokhiemer
Graduate Student

1-Introduction

The description of the interaction between light and matter lies at the heart of physics. A complete theory describing the electrons (matter) must include the interactions with the electromagnetic field (light). The theory has developed many times beginning from the naïve coulomb force between two electrons to the quantum mechanical description of electrons (Schrodinger's equation) and the relativistic quantum mechanical description (Dirac equation) and finally the quantization of the electromagnetic field (quantum electrodynamics- the most accepted theory up to now). In this report the interaction between atoms and electromagnetic fields will be briefly described. In the first part the time dependent perturbation theory will be outlined to describe the interaction using classical fields and quantum theory of matter (semi-classical approach) The interaction Hamiltonian can be obtained using two different approaches. Either by using the minimal coupling or using the multipole expansion and concentrating on the dipole moment as the dominant term. The selection rules for the atomic transitions will be derived using the second approach (dipole approximation). A full quantum analysis of the atom field interaction requires the quantization of the electromagnetic field. So in section three I'll describe the quantization process. At the end I'll use the accepted model of the interaction with quantized field to apply it on the two level atom and see the differences and new results that are predicted by this model. The two level atom will be the platform of applying the theory through the paper.

2-Time-dependent perturbation theory

In this section I'll consider the solution of Schrodinger's equation when the Hamiltonian contains a time dependent part $H'(t)$. The exact form and origin of H' will be described later. Consider a two-level atom having the two states $|a\rangle, |b\rangle$. An atom in a general state can be described as a superposition of the two states: $\Psi(0) = c_a |a\rangle + c_b |b\rangle$.¹ Knowing that the energies corresponding to the two levels are E_a, E_b we can write the time evolution of this state as $\Psi(t) = c_a(t)e^{-iE_a t/\hbar} |a\rangle + c_b(t)e^{-iE_b t/\hbar} |b\rangle$. The normalization of the state function gives $|c_a|^2 + |c_b|^2 = 1$. Note that $|c_a|^2$ is the probability that the atom exists in level $|a\rangle$. The unperturbed time independent Schrodinger equation for both states are $H_o |a\rangle = E_a |a\rangle$,

$H_o |b\rangle = E_b |b\rangle$ Now, applying the Schrodinger's equation to the general wave function using the total

Hamiltonian $H\Psi = i\hbar \frac{\partial\Psi}{\partial t}$ we get

$$c_a [H_o |a\rangle] e^{-iE_a t/\hbar} + c_b [H_o |b\rangle] e^{-iE_b t/\hbar} + c_a [H' |a\rangle] e^{-iE_a t/\hbar} + c_b [H' |b\rangle] e^{-iE_b t/\hbar} =$$

¹ This is (superposition of different states) , in my opinion, one of the main issues causing all the puzzles of quantum mechanics.

$i\hbar \left[\dot{c}_a(t) e^{-iE_a t/\hbar} |a\rangle + \dot{c}_b e^{-iE_b t/\hbar} |b\rangle + c_a \left(-\frac{iE_a}{\hbar} \right) e^{-iE_a t/\hbar} |a\rangle + c_b \left(-\frac{iE_b}{\hbar} \right) e^{-iE_b t/\hbar} |b\rangle \right]$. It's clear that the first two terms cancel the last two terms. We end up by

$c_a [H' |a\rangle] e^{-iE_a t/\hbar} + c_b [H' |b\rangle] e^{-iE_b t/\hbar} = i\hbar \dot{c}_a(t) e^{-iE_a t/\hbar} |a\rangle + \dot{c}_b e^{-iE_b t/\hbar} |b\rangle$. We can isolate \dot{c}_a, \dot{c}_b by taking the inner product with $\langle a|, \langle b|$ respectively and exploiting the orthogonality property : $\langle a|b\rangle=0$

we find after multiplying by $\left(\frac{-i}{\hbar} \right) e^{iE_b t/\hbar}, \left(\frac{-i}{\hbar} \right) e^{iE_a t/\hbar}$

$$\dot{c}_a = \frac{-i}{\hbar} [c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar}], \quad \dot{c}_b = \frac{-i}{\hbar} [c_b H'_{bb} + c_a H'_{ba} e^{-i(E_a - E_b)t/\hbar}]$$

where $H'_{ab} = \langle a|H'|b\rangle$. Usually the diagonal elements H'_{bb}, H'_{aa} vanish, hence

$$\dot{c}_a = \frac{-i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b, \quad \dot{c}_b = \frac{-i}{\hbar} H'_{ba} e^{-i\omega_0 t} c_a \quad \text{where } \omega_0 = \frac{E_b - E_a}{\hbar}.$$

The two equations can be solved iteratively in an approximate manner up to several orders of H' as follows:

To solve up to a first order we assume that the perturbation H' has no effect at all and the atom stays in $|a\rangle$. Thus $c_a^{(0)} = 1, c_b^{(0)} = 0$. We use these values to solve the equations of motion in the next step upto a

first order of H' as follows $c_a^{(1)} = 1, \dot{c}_b = \frac{-i}{\hbar} H'_{ba} e^{-i\omega_0 t}$ Thus, $c_b = \int_0^t \frac{-i}{\hbar} H'_{ba}(t') e^{-i\omega_0 t'} dt'$. The next step is

to use this value and calculate the value of c_a to be $c_a = 1 + \int_0^t \frac{-i}{\hbar} H'_{ab}(t'') e^{-i\omega_0 t''} dt'' \int_0^{t''} \frac{-i}{\hbar} H'_{ba}(t') e^{-i\omega_0 t'} dt'$.

2.1 Sinusoidal perturbation

Suppose that the perturbation Hamiltonian has a sinusoidal time dependence in the form

$H'(r,t) = V(r) \cos(\omega t)$. Thus the matrix elements H'_{ab} are given by $H'_{ab} = V_{ab} \cos(\omega t)$. The first order

approximation for $c_b(t)$ is given by $c_b(t) = \frac{-i}{\hbar} V_{ba} \int_0^t \cos(\omega t') e^{-i\omega_0 t'} dt' = -\frac{V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0 + \omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \right]$

By letting the frequency of the perturbing Hamiltonian be very close to resonant frequency of the two states, the second term dominates the first term. This is called **rotating wave approximation**. We end after eliminating the first term and taking an exponential factor outside by

$c_b(t) = -i \frac{V_{ba}}{\hbar} \frac{\sin[(\omega_0 - \omega)t/2]}{\omega_0 - \omega} e^{i(\omega_0 - \omega)t/2}$. The **transition probability** between the two states is

$P_{a \rightarrow b}(t) = |c_b(t)|^2 \cong \frac{|V_{ba}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$. We notice that the probability varies sinusoidally

between 0 and $\frac{|V_{ba}|^2}{\hbar^2} \frac{1}{(\omega_0 - \omega)^2}$. The system is said to be undergoing flopping in its state between $\langle a |, \langle b |$ by a frequency equal to the difference of the transition frequency and the radiation frequency. This flopping is characteristic even of the exact solution and is not peculiar to the perturbative approximation. An important notice here is that we haven't made any assumptions about the states $\langle a |, \langle b |$ regarding their relative energies. If $E_a > E_b$ the transition from $a \rightarrow b$ stimulated by the radiation is called stimulated emission. Conversely if $E_a < E_b$ the transition is an absorption. Spontaneous emission can't be interpreted using this analysis, but it needs other assumptions as will be shown at the second part of this paper. An important concern here is the question of conservation of energy. We notice that the transition can occur, though by a small probability, even if the frequency of the monochromatic light is different than $\frac{E_a - E_b}{\hbar}$. Doesn't that represent a violation of energy conservation? One possible resolution of this problem is obtained by plotting the transition probability as a function of the radiation frequency keeping the time as a parameter. As shown in figure (1) we see that when the time is very large, the function $\frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$ approaches a delta function and transition will be possible only when the radiation frequency matches the energy difference between the two levels.

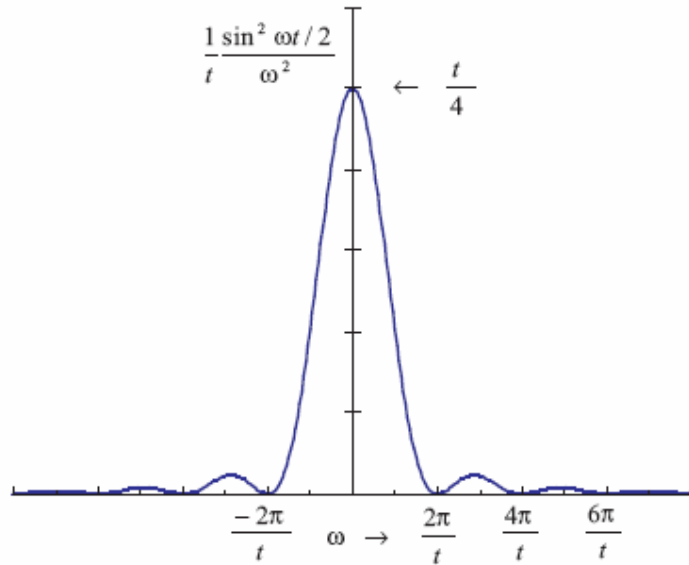


Figure (1)

For very small exposure time, we notice that the transition probability has non-zero values for $\omega \neq \omega_0$. Yes this is a violation of the conservation of energy, but it happens for a very short interval. When a transition happens at $\omega \neq \omega_0$, and if the radiation is still existing, the radiation will continue to perturb

the atom (as can be seen when resolving the problem with the new initial conditions) and the electron will eventually go back to the other level. This is in conform with the uncertainty principle. Taking the difference between ω, ω_0 to be inversely proportional to (t) , we find immediately that $\Delta E t \sim \hbar$. A more critical question arises: Can this momentarily violation of the conservation of energy have any observable effects? To answer this question I propose the three-level atom as a test bench for another problem. Consider the energy spectrum of an atom show in figure (2).

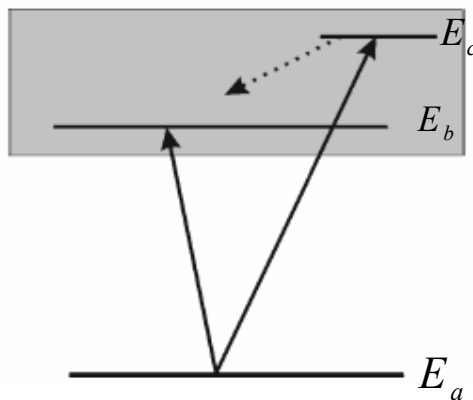


Figure (2)

Suppose that the incident of frequency ω doesn't match any of the transition frequencies inside the atom. Let the atom be initially in the ground state $|a\rangle$ and the incident radiation has perturbed the atom to jump to level $|c\rangle$. Now the atom can decay through two different routes, directly to $|a\rangle$ or to $|b\rangle$ then to $|a\rangle$. In the second route the atom will emit two photons of frequencies different than ω . The detection of these two photons is a direct detection of the violation of energy conservation !

2.2 The interaction Hamiltonian

In the previous treatment we used the interaction Hamiltonian without knowing its value. Now it's the time to inspect how is the interaction Hamiltonian calculated. From classical mechanics, the generalized momentum that appears in the Lagrangian doesn't necessarily have to be the physical momentum which is the amount of motion. In the classical interaction between charges and fields, the generalized momentum is the usual momentum added to it a term proportional to the magnetic vector potential. Hence the total energy which is the classical Hamiltonian in this case is given by

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi(r) + V(r)$$

where \vec{p} is the generalized momentum and hence the first term

describes the kinetic energy and $q\phi$ is the electric potential of the EM radiation while $V(r)$ is the Coulomb potential between the nucleus and the electron. The first term is the kinetic energy and the second term is the electric potential energy. The magnetic field doesn't contribute to the energy of the particle (at least directly). Going to the quantum mechanics of the electron interaction with the EM field

and using the correspondence principle, we use the same form of the Hamiltonian. This is called **minimal coupling**. We see that the Hamiltonian will have two new additional terms corresponding to the interaction, namely: $H = \frac{1}{2m} p^2 + qV - \frac{q}{2m} (\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}) + \frac{q^2}{2m} A^2 + q\phi$. The new generalized momentum still have the same operator representation, $-i\hbar\vec{\nabla}$. The term containing A^2 leads to a small correction to the energies of the states and will therefore be neglected. We notice that it is its own operator (just a multiplicative factor) and hence its contribution to the transition probability is zero ($\langle a|A^2|b\rangle = 0$). As for the $q\phi$ term, it's taken to be zero under the Coulomb gauge $\nabla \cdot \vec{A} = 0$. So the interaction energy here contributes in the kinetic energy and not the potential energy of the electron. I think it's logical to consider the effect of the interaction energy in either of the kinetic or potential energies not both. Let's investigate the commutation relation between the momentum and the magnetic vector potential to see how does the third term contribute:

Using the definition of the momentum operator it's easy to show that $[p_k, A_j] = -i\hbar \frac{\partial A_j}{\partial x_k}$ as follows

$$\begin{aligned} [p_k, A_j] \psi(\vec{r}) &= -i\hbar (\partial_k A_j - A_j \partial_k) \psi(\vec{r}) = -i\hbar \partial_k (A_j \psi) + i\hbar A_j \partial_k \psi \\ &= -i\hbar (\partial_k A_j) \psi - i\hbar A_j (\partial_k \psi) + i\hbar A_j (\partial_k \psi) = -i\hbar (\partial_k A_j) \psi. \end{aligned}$$

And since $(\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p})$ can be written as $[p_k, A_k] + 2p_k A_k$ which is equivalent from the previous result to $-i\hbar \vec{\nabla} \cdot \vec{A} + 2\vec{A} \cdot \vec{p}$. Since the magnetic vector potential is arbitrary up to the gradient of an arbitrary gauge function, we can choose to work in the Coulomb gauge and set $\vec{\nabla} \cdot \vec{A} = 0$.

Let the vector potential $\vec{A} = A_0 \vec{\epsilon} \cos(\vec{k} \cdot \vec{x} - \omega t)$ where $\vec{\epsilon}$ is the polarization vector and $\vec{\epsilon} \cdot \vec{k} = 0$.

Hence $\vec{A} \cdot \vec{p} = A_0 \vec{\epsilon} \cdot \vec{p} \cos(\vec{k} \cdot \vec{x} - \omega t)$. Assuming the wavelength is long enough compared to the dimensions of the atom, we can neglect $\vec{k} \cdot \vec{x}$ (this is called dipole approximation), and invoking the result developed early in this section, we find that the transition probability changes sinusoidally with time and its amplitude is proportional to $A_0^2 [\vec{\epsilon} \cdot \langle b | \vec{p} | a \rangle]^2$. Assuming the polarization vector in the x-

direction and using the fact that $[x, H_0] = i\hbar p_x / m$ then it follows that $\langle b | p_x | a \rangle = \frac{im}{\hbar} (E_b - E_a) \langle b | x | a \rangle$.

This is the main result of this part that we will invoke later on to calculate the selection rules governing the transition between different states. Note that the field strength of the incident radiation is assumed to be so weak that the energies of the stationary levels are assumed to be unchanged by the perturbing radiation.

The interaction Hamiltonian can be obtained in another way using multipole expansion of the atomic charge and calculating the interaction with the electric field of the external radiation.

$$\underbrace{W}_{\text{field-atom interaction energy}} = \underbrace{V_0 \int \rho(x) \cdot dx}_{\text{netcharge on atom}} + \underbrace{\left[\frac{\partial V}{\partial x} \right]_0 \cdot \int \rho(x) \cdot x \cdot dx}_{\substack{\text{E-field at the origin} \\ \text{electric dipole moment of the atom}}} + \underbrace{\frac{1}{2!} \left[\frac{\partial^2 V}{\partial x^2} \right]_0 \cdot \int \rho(x) \cdot x^2 \cdot dx}_{\substack{\text{E-field gradient at origin} \\ \text{electric quadrupole moment of the atom}}}$$

By noticing that the dipole term is much larger than the higher order terms, the dipole term alone can be retained and this is the more formal way to define the **dipole approximation**. The reason behind this approximation is that the field is assumed to be of the same value through the atom at the same instant and hence its gradient vanishes. It's claimed that the two interaction Hamiltonian are equivalent.² The interaction Hamiltonian of the dipole approximation is given by $-\vec{d} \cdot \vec{E}$ where \vec{d} is the operator corresponding to the dipole moment of the atom ($\vec{d} = e\vec{r}$) and \vec{E} is the electric field at the location of $\vec{r} = 0$. We note here that this is an electrostatic expression coming from the integral of

$$-\sum_i^{all} q_i \int_{\vec{r}_{ref}}^{\vec{r}_i} \vec{E}_T(\vec{r}) \cdot d\vec{r}$$

The magnetic dipole interaction is of smaller value since from Maxwell's equations $B = \nabla \times A$ and A is assumed to have no spatial variation inside the atom. We find again that the transition amplitude will be proportional to $\langle b | r | a \rangle$.

2.3 Selection rules for atomic transitions

In deriving the transition rules governing the quantum jumps, we will require that $\langle b | r | a \rangle \neq 0$. Letting the states $|a\rangle, |b\rangle$ be characterized by the quantum numbers $|n, l, m\rangle, |n', l', m'\rangle$ respectively then any of the relations $\langle n, l, m | x | n', l', m' \rangle \neq 0, \langle n, l, m | y | n', l', m' \rangle \neq 0, \langle n, l, m | z | n', l', m' \rangle \neq 0$ will be a selection rule. Using the commutation relations between the angular momentum operator and the position operator $[L_z, x] = i\hbar y, [L_z, y] = -i\hbar x, [L_z, z] = 0$ we deduce that $0 = \langle n, l, m | [L_z, z] | n', l', m' \rangle = \langle n, l, m | (L_z z - z L_z) | n', l', m' \rangle = \langle n, l, m | (m\hbar z - z m' \hbar) | n', l', m' \rangle = (m - m')\hbar \langle n, l, m | z | n', l', m' \rangle$. Then for $\langle n, l, m | z | n', l', m' \rangle \neq 0$ m and m' have to be equal. Meanwhile from the commutation relation between $[L_z, x]$ we find

² See for example "Equivalence of interaction Hamiltonians in the electric dipole approximation", journal of modern optics, vol. 51, no. 8, 1137–1147

$\langle n, l, m | [L_z, x] | n', l', m' \rangle = \langle n, l, m | (L_z x - x L_z) | n', l', m' \rangle = \langle n, l, m | (m \hbar x - x m' \hbar) | n', l', m' \rangle$ and hence $(m - m') \hbar \langle n, l, m | x | n', l', m' \rangle = i \hbar \langle n, l, m | y | n', l', m' \rangle$.

Similarly we can prove that $(m - m') \hbar \langle n, l, m | y | n', l', m' \rangle = -i \hbar \langle n, l, m | x | n', l', m' \rangle$.

By combining these two equations we get $(m - m')^2 \langle n, l, m | x | n', l', m' \rangle = \langle n, l, m | x | n', l', m' \rangle$.

Again for $\langle n, l, m | x | n', l', m' \rangle \neq 0$ we have to set the difference between m, m' to be ± 1 . We conclude that **no transitions can occur unless $\Delta m = \pm 1$ or 0** . This result could have been obtained by writing the matrix elements as integrals of the spherical harmonics $Y_l^m(\theta, \phi)$ and selecting the values of m, m' that make the integral $\int Y_l^m(\theta, \phi) f(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) d\Omega \neq 0$. Where $f(\theta, \phi)$ describes the angular dependence

of $-\vec{d} \cdot \vec{E}$. To derive the selection rules of the orbital angular momentum quantum number l, l' we will make use of the commutation relation $[L^2, [L^2, r]] = 2\hbar^2(rL^2 + L^2r)$ which has the

value $2\hbar^4(l'(l'+1) + l(l+1)) \langle n, l, m | r | n', l', m' \rangle$. Since $\langle n, l, m | [[L^2, [L^2, r]] | n', l', m' \rangle$

$= \hbar^2(l'(l'+1) - l(l+1)) \langle n, l, m | [L^2, r] | n', l', m' \rangle$ which in turn =

$\hbar^4(l'(l'+1) - l(l+1))^2 \langle n, l, m | r | n', l', m' \rangle$ we conclude that $2(l'(l'+1) + l(l+1)) = (l'(l'+1) - l(l+1))^2$.

After some algebraic manipulation this condition can be written as $[(l'+l+1)^2 - 1][(l'-1)^2 - 1] = 0$

which has simple solution: either $|l' - l| = 1$ or $l' = l = 0$. The second possibility is excluded since the

angular integral will have the dependence on the θ contained in $-\vec{d} \cdot \vec{E}$ which will cancel out when integrated over θ when choosing the z-direction in the direction of E. Then we are left with the **second selection rule $\Delta l = \pm 1$** .

We notice in the rules that the total angular momentum can change between the different states. The intrinsic angular momentum of the emitted or absorbed photon (spin) is believed to carry this difference.

This is not the whole story. In cases where the electric dipole transitions are forbidden higher order transitions may occur with different selection rules. In electric quadrupole transition, the atom interacts with the EM field as an electric quadrupole not as a dipole. Similarly in magnetic dipole transition the atom interacts as a magnetic dipole since the atom possesses a magnetic dipole from the angular momentum and spin of its electrons. In this case the matrix element term of interest would be

$\langle n, l, m | L | n', l', m' \rangle$ where L is the angular momentum or spin of interest instead of $\langle n, l, m | r | n', l', m' \rangle$.

Note that this treatment is unable to predict the properties of the emitted photon in case of stimulated emission and can't prove the coherence between the incident photon and the emitted one.

3- Quantization of the electromagnetic field

Up to now, we have treated the electromagnetic field classically without doing any quantization. The modern theory of electrodynamics, Quantum Electrodynamics, treats the field modes as quantized harmonic oscillators. One major difference that will be predicted by this treatment is the possibility of transitions even when there are no external radiation at all in a process called spontaneous emission. The reason for that is that the energy of the ground state of the quantum harmonic oscillator is not zero but there is always quantum fluctuations corresponding to the zero point energy.

In what follows I'll show how the electromagnetic field inside a cavity of volume V is quantized. Beginning with Maxwell's field equations in free space we find that the vector potential $A(\vec{r},t)$ satisfies the homogenous wave equation :

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta\right) \vec{A}(\vec{r},t) = 0$$

Expanding $A(\vec{r},t)$ into a complete set of wave functions $\vec{u}_{\vec{k},\sigma}(\vec{r})$

$$\vec{A}(\vec{r},t) = \sum_l A_l q_l(t) u_l(\vec{r})$$

which as we will prove later obey the orthogonality relation $\frac{1}{\sqrt{V_l V_l'}} \int u_l(\vec{r}) u_{l'}(\vec{r}) d\vec{r}^3 = \delta_{l,l'}$,

Assuming that the vector potential obeys the Coulomb gauge : $\nabla \cdot A = 0$ we get

$$0 = \text{div } \vec{A}(\vec{r},t) = \sqrt{\frac{1}{\epsilon_0}} \sum_n q_n(t) \text{div } \vec{u}_n(\vec{r})$$

Since this relation is valid at all instances of t , we conclude that $\text{div } u_n(\vec{r}) = 0$.

By applying this expansion in the wave equation of the vector potential we get

$$\frac{1}{c^2} \ddot{q}_n(t) \vec{u}_n(\vec{r}) - q_n(t) \nabla^2 \vec{u}_n(\vec{r}) = 0.$$

By adding a term of magnitude $\frac{\omega_n^2}{c^2} q_n(t) u_n(\vec{r})$ and subtracting it we obtain

$$\frac{1}{c^2} \left[\ddot{q}_n(t) + \omega_n^2 q_n(t) \right] \vec{u}_n(\vec{r}) - q_n(t) \left[\nabla^2 \vec{u}_n(\vec{r}) + \frac{\omega_n^2}{c^2} \vec{u}_n(\vec{r}) \right] = 0.$$

Similar to separation of variables we can divide the equation by $q_n(t) u_n(\vec{r})$ and see that

$$\begin{aligned} \ddot{q}_n(t) + \omega_n^2 q_n(t) &= \text{const } q_n(t) \\ \nabla^2 \vec{u}_n(\vec{r}) + \frac{\omega_n^2}{c^2} \vec{u}_n(\vec{r}) &= \text{const } u_n(\vec{r}). \end{aligned}$$

Then we can redefine ω_n to include the constant and get

$$\begin{aligned} \ddot{q}_n(t) + \omega_n^2 q_n(t) &= 0, \\ \nabla^2 \vec{u}_n(\vec{r}) + \frac{\omega_n^2}{c^2} \vec{u}_n(\vec{r}) &= 0. \end{aligned}$$

We note that the equation of $q_n(t)$ is that of a Harmonic Oscillator equation. Expressing the electric and magnetic fields in terms of these basis functions we

$$E = -\frac{\partial}{\partial t} \vec{A}(\vec{r}, t) = -\sum A_l \dot{q}_l(t) \vec{u}_l(\vec{r})$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \sum A_l q_l(t) \nabla \times \vec{u}_l(\vec{r})$$

The total energy inside the resonator is $\mathcal{E} = \int \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) dr^3$

By substituting with the field expressions we get

$$\mathcal{E} = \frac{1}{2} \int_V d\vec{r} \left[\sum_{n,m} \dot{q}_m(t) \dot{q}_n(t) \vec{u}_m(\vec{r}) \cdot \vec{u}_n(\vec{r}) + \frac{c^2}{2} \sum_{m,n} q_m(t) q_n(t) \text{rot } \vec{u}_m(\vec{r}) \cdot \text{rot } \vec{u}_n(\vec{r}) \right].$$

Consider the second integral which we call J_{mn}

$$J_{mn} = \int_V d\vec{r} \text{rot } \vec{u}_m(\vec{r}) \cdot \text{rot } \vec{u}_n(\vec{r}),$$

From the vector analysis identities

$$\begin{aligned} \text{rot } \vec{a} \cdot \text{rot } \vec{b} &= \text{div} \left(\vec{a} \times \text{rot } \vec{b} \right) + \vec{a} \cdot \text{rot rot } \vec{b} \\ &= \text{div} \left(\vec{a} \times \text{rot } \vec{b} \right) + \vec{a} \cdot \text{grad div } \vec{b} - \vec{a} \cdot \nabla^2 \vec{b}, \end{aligned}$$

Then we find that J_{mn} can be written as

$$J_{mn} = \int_V d\vec{r} \left[\text{div} \left(\vec{u}_m \times \text{rot } \vec{u}_n \right) + \vec{u}_m \cdot \text{grad div } \vec{u}_n - \vec{u}_m \cdot \nabla^2 \vec{u}_n \right].$$

The second term of J_{mn} in the integral vanishes from the gauge condition and we are left with

$$J_{mn} = \int_V d\vec{r} \left[\text{div} \left(\vec{u}_m \times \text{rot } \vec{u}_n \right) + \frac{\omega_n^2}{c^2} \vec{u}_m \cdot \vec{u}_n \right].$$

where use has been made of the wave equation of $u_n(\vec{r})$. From the symmetry of J_{mn} , i.e.,

$J_{mn} - J_{nm} = 0$ we get

$$\int_V d\vec{r} \left[\text{div} \left(\vec{u}_m \times \text{rot } \vec{u}_n \right) - \text{div} \left(\vec{u}_n \times \text{rot } \vec{u}_m \right) \right] = \frac{\omega_m^2 - \omega_n^2}{c^2} \int_V d\vec{r} \vec{u}_m \cdot \vec{u}_n,$$

From Gauss theorem we convert the volume integral in the LHS into a surface integral that is we have

$$\int_V d\vec{r} \text{div} \left(\vec{u}_m \times \text{rot } \vec{u}_n \right) = \oint_{\partial V} d\vec{S} \cdot \left(\vec{u}_m \times \text{rot } \vec{u}_n \right) = 0.$$

From the boundary conditions inside the cavity³, the tangent component of the electric field should

vanish on the walls as well as the normal component of the magnetic field. Hence $\vec{u}_n(\vec{r})|_{tangent} = 0$

And since the component of u_m that contribute in the cross product is the tangential component only we

$$\frac{\omega_m^2 - \omega_n^2}{c^2} \int_V d\vec{r} \vec{u}_m \cdot \vec{u}_n = 0.$$

see that this integral will vanish and consequently

This provides the orthogonality relation of u_m, u_n which after normalized can be written as

$$\int_V d\vec{r} \vec{u}_m \cdot \vec{u}_n = \delta_{mn},$$

Since we proved that the first term in J_{mn} vanishes, then J_{mn} can be written as

$$J_{mn} = J_{nm} = \int_V d\vec{r} \text{rot } \vec{u}_m(\vec{r}) \cdot \text{rot } \vec{u}_n(\vec{r}) = \frac{\omega_n^2}{c^2} \delta_{mn}.$$

By plugging this value into the expression of the energy we get

$$\mathcal{E} = \frac{1}{2} \sum_n \dot{q}_n^2(t) + \frac{c^2}{2} \sum_{m,n} q_m(t)q_n(t) \frac{\omega_n^2}{c^2} \delta_{mn} = \frac{1}{2} \sum_n \left(\dot{q}_n^2(t) + \omega_n^2 q_n^2(t) \right).$$

Since the Hamiltonian consists of generalized coordinates and generalized momenta, we define $q(t)$ to be the generalized coordinate and $\dot{q}_n(t)$ to be the generalized momentum. We note that Hamilton's

equations when applied to this Hamiltonian lead directly to $\ddot{q}_n(t) + \omega_n^2 q_n(t) = 0$, which was derived earlier. From the theory of the quantum harmonic oscillator, we define the momentum and

coordinate operator such that they satisfy the commutation relation $[\hat{q}_l, \hat{p}_l] = i\hbar\delta_{l,l'}$, and the

Hamiltonian operator is written in terms of these operators as $\hat{H} = \sum \hat{H}_l = \sum \frac{1}{2} \hat{p}_l^2 + \frac{1}{2} \omega_l^2 \hat{q}_l^2$

Note that all the following discussion is in complete analogy with the quantum harmonic oscillator. We define the lowering and raising as

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega q + ip), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}} (\omega q - ip).$$

the operator a_n^+ adds one photon of mode n to the field while the operator a_n destroys one photon of mode n.

³ I don't know why we are quantizing inside a cavity while the field exists in free space !

The operators corresponding to the electromagnetic vector fields and potential are

$$\begin{aligned}\vec{A}_\perp(\vec{r}, t) &= \sum_n \sqrt{\frac{\hbar}{2\epsilon_0 \omega_n}} (\hat{a}_n + \hat{a}_n^\dagger) \vec{u}_n(\vec{r}), \\ \vec{E}_\perp(\vec{r}, t) &= i \sum_n \sqrt{\frac{\hbar \omega_n}{2\epsilon_0}} (\hat{a}_n - \hat{a}_n^\dagger) \vec{u}_n(\vec{r}), \\ \vec{B}(\vec{r}, t) &= \sum_n \sqrt{\frac{\hbar}{2\epsilon_0 \omega_n}} (\hat{a}_n + \hat{a}_n^\dagger) \text{rot } \vec{u}_n(\vec{r})\end{aligned}$$

Note that field operators don't commute, so E and B can't be measured simultaneously upto arbitrary accuracies.

To determine the operator corresponding to the Hamiltonian determined previously we notice that

$$a_n^\dagger a_n = \frac{1}{2\hbar\omega} [\omega_n^2 \hat{q}_n^2 + \hat{p}_n^2 + i\omega_n (\hat{q}_n \hat{p}_n - \hat{p}_n \hat{q}_n)] \text{ and from the commutation relation } [\hat{q}_l, \hat{p}_{l'}] = i\hbar \delta_{l,l'}, \text{ we}$$

get $\frac{1}{2}(\omega_n^2 \hat{q}_n^2 + \hat{p}_n^2) = \hbar\omega_n \left(a_n^\dagger a_n + \frac{1}{2} \right)$. Thus the Hamiltonian operator is given

by $H = \sum_n \hbar\omega_n \left(a_n^\dagger a_n + \frac{1}{2} \right)$. Note that the index n is actually a multi-index symbol that represents

different wave numbers and polarization directions. A more explicit form is

$$H_{\text{fields}} = \sum_{k,\lambda} \left(a_{k,\lambda}^\dagger a_{k,\lambda} + \frac{1}{2} \right) \hbar\omega.$$

When this operator is applied on a state containing n photons $|n\rangle$ we get

$$H_{\text{osc}} |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle$$

We note that when n=0 the energy is not zero and this is the zero point energy referred to at the beginning of this section.

4- Interaction between atoms and quantized fields

Now to study the interaction between an atom and an electromagnetic field assume as before that the

atom consists of two levels $\langle g |, \langle e |$ and let the electric field operator be $\hat{E} = \vec{e} \left(\frac{\hbar\omega}{\epsilon_0 V} \right)^{1/2} (\hat{a} + \hat{a}^\dagger) \sin(kz)$ ⁴.

Where the spatial dependence is contained in $\sin(kz)$ while the time dependence is in the raising and the lowering operators.

Define the atomic transition operators as The two states are assumed to have opposite parity (hence dipole transition is allowed) and orthogonal to each other. Since the basis $\langle g |, \langle e |$ form a complete set,

⁴ According to the previous field operators the plus sign between a, a^\dagger should be a minus sign, but I couldn't understand under what assumptions was this plus used instead.

any operator can be represented as $\hat{O} = O_{gg}\hat{\sigma}_{gg} + O_{ge}\hat{\sigma}_{ge} + O_{eg}\hat{\sigma}_{eg} + O_{ee}\hat{\sigma}_{ee}$ where $O_{ij} = \langle i|\hat{O}|j\rangle$. and

$\hat{\sigma}_{ij} = |i\rangle\langle j|$, in particular the operator can be represented as $\hat{\mathbf{d}} = \mathbf{d}_{ge}\hat{\sigma}_{ge} + \mathbf{d}_{eg}\hat{\sigma}_{eg}$ where the property that the states $\langle g|, \langle e|$ have opposite parities lead to the vanishing of the other two terms.

$\hat{\sigma}_+ = |e\rangle\langle g|$ (excitation operator)

$\hat{\sigma}_- = |g\rangle\langle e|$ (decay operator)

$\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|$ (atomic transition operator)

The dipole operator is $\mathbf{d}(\hat{\sigma}_- + \hat{\sigma}_+)$

If we define the zero energy level to be halfway between $\langle e|, \langle g|$ and expressed the unperturbed Hamiltonian of the atom as a two-term dyad (in terms of the eigenstates) we have

$$\hat{H}_{Atom} = \frac{1}{2}(E_e - E_g)\hat{\sigma}_3 = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3. \text{ Now, using the usual dipole approximation } \hat{H}_{Interaction} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$$

$$\therefore \hat{H}_{Interaction} = \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^+) \text{ where } \lambda = -d \left(\frac{\hbar\omega}{\epsilon V} \right)^{1/2} \sin(kz) / \hbar.$$

The time dependence of the operators is given by $\hat{a}(t) = \hat{a}(0)e^{-i\omega t}, \hat{a}^+(t) = \hat{a}^+(0)e^{+i\omega t}, \hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega_0 t}$

Hence $\hat{\sigma}_+\hat{a} \sim e^{i(\omega_0 - \omega)t}, \hat{\sigma}_+\hat{a}^+ \sim e^{i(\omega_0 + \omega)t}, \hat{\sigma}_-\hat{a}^+ \sim e^{-i(\omega_0 - \omega)t}, \hat{\sigma}_-\hat{a} \sim e^{-i(\omega_0 + \omega)t}$. When the last two terms are

used in integrating Schrodinger's equation they will bring $(\omega_0 + \omega)$ in the denominator and hence will be dominated by the first two terms. By doing the rotating wave approximation again we can neglect them

and the interaction Hamiltonian will be $\hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^+)$. This is called the **Jaynes-Cumming model** for

the atom field interaction. Now let's try to analyze two simple cases using this model. For the first case assume that the field consists of a single mode containing n photons. Suppose that the atom initially is in the excited state $|e\rangle$, thus the total system is in the initial state $|i\rangle = |e\rangle|n\rangle$. The only final state can be

$|f\rangle = |g\rangle|n+1\rangle$. A general state vector is given as a superposition of these two states

$|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle$. Now let's apply Schrödinger's equation using the interaction Hamiltonian

alone since it's the only part in the Hamiltonian that causes the time evolution. By substituting in

Schrödinger's equation $i\hbar \frac{d|\psi(t)\rangle}{dt} = \hat{H}_{int}|\psi(t)\rangle$ we get the following coupled equations of motion

$$\dot{C}_i = -i\lambda\sqrt{n+1}C_f, \dot{C}_f = -i\lambda\sqrt{n+1}C_i. \text{ Solving them simultaneously we get } \ddot{C}_i + \lambda^2(n+1)C_i = 0. \text{ The}$$

solution matching the initial conditions is $C_i(t) = \cos(\lambda t\sqrt{n+1}), C_f(t) = -i\sin(\lambda t\sqrt{n+1})$. The

probability that the system makes a transition to the final state is $P_f(t) = C_f^2(t) = \sin^2(\lambda t\sqrt{n+1})$.

Atomic inversion is given by $P_f(t) - P_i(t) = \cos(2\lambda t\sqrt{n+1})$. The oscillation between the final and

initial states are called Rabi oscillation and they are analogous to the classical oscillations shown in section 2. However the oscillations are present even when there is no photons and they are called vacuum Rabi oscillations. The quantization of the electromagnetic field implies the existence of new and strange states of light according to the photon distribution in these states. A more general field state is given as $\sum C_n(t)|n\rangle$. The atomic inversion in this case is given by $\sum |C_n(t)|^2 \cos(2\lambda t \sqrt{n+1})$. An interesting phenomenon occurs when the field state is the coherent state in which the coefficients C_n are

given by $C_n = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}$. The inversion in this case will be $W(t) = e^{-\bar{n}} \sum \frac{\bar{n}^n}{n!} \cos(2\lambda t \sqrt{n+1})$ where

$\bar{n} = \langle a^\dagger a \rangle$ is the mean photon number. This is a non-classical effect where the Rabi oscillation collapse and then after sometime revive again as shown in figure (3).

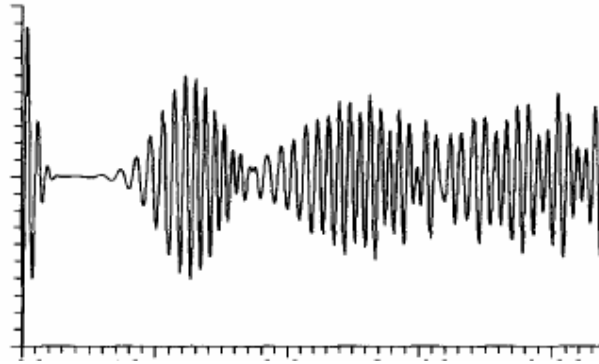


Figure (3)

5- Conclusion

The interaction between matter and light has been introduced from a semi-classical approach and a full quantum approach. The main step in both approaches is to obtain the correct form of the interaction Hamiltonian. Useful approximations were utilized in both methods, mainly the rotating wave approximation and dipole approximation. The selection rules for the dipole approximation have been derived which show that the angular momentum of the electron has to change by ± 1 conforming with the fact that the photons are spin-1 bosons. The process of quantizing the electromagnetic field inside a conducting cavity has been described in detail. The new phenomena predicted by the quantized field interaction with the atom that are not predicted by the semi-classical analysis can provide new tests for the quantized field model.

6- References

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