

Relativistic Electrodynamics

1-Introduction and review of special relativity theory

During the nineteenth century the theory of electromagnetism was established and crowned by the Maxwell's equations which associated electromagnetism with wave phenomenon and light. This formulation of the electromagnetic phenomenon seemed to be plausible except for one thing. Known wave phenomenon such as sound waves and other elastic wave required a medium of propagation. Such an analog was not existing yet for the new electromagnetic waves. For this reason the luminiferous ether was postulated to exist all over space to work as a medium for the propagation of the EM waves. The problem that accompanied the wave phenomenon of the EM fields is its invariance under Galilean transformation. Galilean transformation is the transformation of the coordinates between two reference frames in uniform motion with respect to each other. Assuming that one frame of reference K moves by a velocity V with respect to another frame K' in the x direction, we find that the coordinates and velocities in both frames are related by the transformation relations:

$$x = x' + Vt$$

$$v = v' + V$$

$$t = t'$$

$$\nabla = \nabla'$$

Using these transformations, we find that the laws of classical mechanics take the same form in both frames. For example, if we have a system of interacting particle via two

body interactions V_{ij} , we find that the equation of motion $m \frac{d^2 x_i}{dt^2} = -\nabla \sum_j V_{ij} |x_i - x_j|$

have the same form in K' namely: $m \frac{d^2 x'_i}{dt'^2} = -\nabla' \sum_j V_{ij} |x'_i - x'_j|$. Being second order in

both time and space coordinates we find that the wave equation is not invariant under such transformation. That was attributed to the nature of the wave propagation that require a transmitting medium and hence the wave equation is valid only in the frame of the medium. Thus the ether was viewed to play the same rule in electromagnetics that the air plays in sound waves though the apparent difference between the nature of the wave in both phenomena; the first consists of field oscillations while the second is mechanical vibrations. Attempts soon began to measure the speed of earth w.r.t the ether frame of reference. The most famous experiment in this regard is the Michelson-Morley experiment which yielded the most famous null result in physics. Many assumptions were assumed to justify the null results of these experiments such as the length contraction hypothesis and the Ritz's emitter theory. Experiments trying to repeat the original ones with much higher accuracy didn't stop up to this moment¹. The results of most of the experiments are consistent with special relativity. Einstein formulated his relativity theory based on two postulates:

Postulate 1: All the laws of physics are the same in all inertial frames of reference.

Postulate 2: The speed of light is finite and independent of the motion of the source.

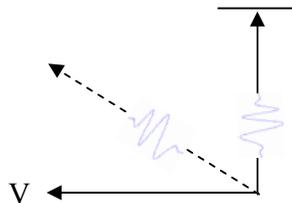
¹ Some of the most recent experiments are:

- 1- Trying to test the time dilation consequence of special relativity by sending a precise clock on board of future satellites
- 2- Trying to repeat Kennedy-Thorndike experiment (a modified version of Michelson-Morley experiment) using cryogenic cooled cavity lasers

The first postulate was known for classical mechanics a long time ago and its validity for electromagnetic was the objective of the previous experiments. Einstein generalized it for all physical laws. It means that if two observers performed an experiment in 2 different reference frames moving with constant relative velocity (inertial frames) they will get exactly the same results. In another way, one can not perform an experiment in his own frame of reference that enables him to determine his speed with respect to another frame of reference without referring to other frame. This postulate has omitted the notion of absolute motion since according to it there is no absolute reference frame, all inertial frames are equivalent and have the same physics. Since Maxwell's laws have the same form in any inertial frame with the same constants, the speed of electromagnetic waves i.e, light should be the same in all frames regardless of the speed of the source and this is the second postulate. This postulate modifies the set of transformation under which physical laws are invariant to include a ratio between the velocity of the frame and the speed of light, thus making physical laws invariant under Lorentz transformation rather than Galilean transformation as will be shown later. One of the most astonishing consequences of this theory is concept of relativity of simultaneity. Two events that happen simultaneously in one frame will not be simultaneous in another frame of reference. To illustrate this principle imagine this very simple example: a source moving with velocity v and emitting a light pulse in a direction normal to its motion. In the frame of the source, the pulse will be traveling upwards by a velocity C , while in the frame of a stationary observer; the pulse will be traveling in an inclined direction by a velocity C . This velocity has a horizontal component v , and a vertical component $\sqrt{C^2 - v^2}$. When the pulse reaches a certain height (this corresponds to an event that happens only once for both frames), the moving source will measure a smaller interval of elapsed time from the moment of firing the pulse since it moves with a larger vertical velocity than the stationary observer. In other words, the stationary observer will feel that his time is running slower than the moving observer. This is called time dilation. In other words, if we imagined a hypothetical clock that measures time absolutely without being affected by motion it will count extra time in the moving frame between the same 2 events than a stationary frame.² By simple algebra it can be shown that the relation between the time measured

by the moving observer t and the stationary one t_0 is : $t_0 = \frac{t}{\sqrt{1 - v^2/c^2}}$. Now suppose

that the moving source emits two opposite pulses in two different directions as shown in figure (2), the pulses will be moving with the same velocity w.r.t both frames. However if there are 2 points A,B at fixed positions in the stationary frames they will not be fixed at the source frame so the 2 pulses will reach them simultaneously in the first frame and not simultaneously in the 2nd frame. The fact that the time is running slower in the moving frame while the speed of light is kept constant requires that the length scale in the moving frame be shorter too by the same ratio, this is called length contraction. Keeping these two facts in mind we find that the coordinate transformation between the two frames will not follow the Galilean transformations relations but the Lorentz relations.



² One may argue here that the moving frame may consider itself stationary and the other traveling and thus its clock should count less time, but another one may reply that such clock doesn't exist altogether !!

Fig. 1 A source of light is traveling to the left with respect to some stationary frame.

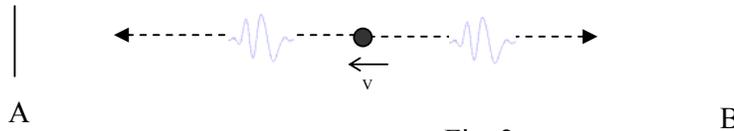


Fig. 2.

Note the interrelation between the two postulates: the first postulate enforces the constancy of the speed of light for Maxwell's equations to be invariant and from the second postulate, Lorentz transformations are generated and when applied on Maxwell's equations they show the sought invariance. Using the length contraction concept we can show that the Lorentz transformation relations between the coordinate systems of two inertial frames in relative motion in the x-axis direction by velocity v as shown in figure (3) are given by :

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right); \text{ where } \gamma \text{ is given by } \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

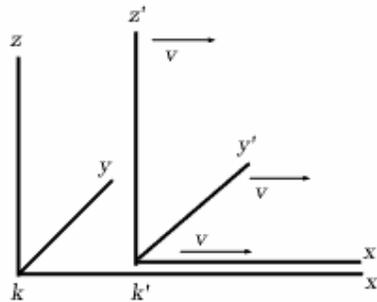


Fig. 2

A direct result of these transformation relations is the relativistic velocity addition rule. If an object is moving in the frame K' by velocity $v' = dx'/dt'$, it can be shown by direct manipulation of the previous equations that the velocity in the same object in frame K is $\frac{v + v'}{1 + \frac{vv'}{c^2}}$. For a relative velocity in a general direction v , the transformation of a general

spatial vector r is done by dividing it into 2 parts normal and parallel to the velocity vector, for which the first is the same in both frames, and the other will be transformed as in the previous equation. As for the spatial coordinates, they can be divided into

$$\vec{r}' = \vec{r}_\perp + \gamma(\vec{r}_\parallel - \vec{v}t)$$

$$t' = \gamma\left(t - \frac{\vec{r} \cdot \vec{v}}{c^2}\right)$$

Lorentz transformation can be written in matrix form as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}.$$

Or equivalently $(x^\mu)' = \sum_{\nu=0}^3 (\Lambda^\mu_\nu) x^\nu$ where Λ^μ_ν are the elements of the transformation

matrix. In the first case where the frame K' was moving in the x-axis direction, it can be shown that the transformation equation are equivalent to a rotation of the x, ict axis by an angle $i\Phi$ where $\Phi = v/c$. It can be shown also that the norm squared of the vector $x^\mu = (ct, x)$ defined as $(ct)^2 - x^2$ is the same in both frames. If we considered the conservation of momentum in 2 different inertial frames using the new velocity addition rules we may discover that the usual expression of the momentum $u=mv$ using the invariant mass m (the same in the two frames) leads to inconsistency. To resolve that inconsistency, the mass was assumed to be a relative quantity and affected as well by the

same transformation $m' = \frac{m}{\sqrt{1-v^2/c^2}}$. The mass in this consideration is considered to

constitute part of the energy of the system (rest energy). Hence, the total energy now constitutes both the kinetic energy and the rest energy : $E^2 = p^2 c^2 + m^2 c^4$.

2. Relativistic electrodynamics

2.1 Basic relations

All experiments that tried to measure the ratio of electron charge to its mass proved that the electric charge of the electron is an invariant quantity. The reason behind this invariance is still unknown, and the only proof is still experimental. Beginning from that invariance of charge let's see how electric and magnetic fields will transform between different inertial frames.

Imagine 2 inertial frames K, K' in relative velocity v and a single charge q stands still in the first frame. We know that Maxwell's equations apply in both frames, and hence we expect the application of Gauss law in its integral form to be verified in both frames. So let's apply Gauss law on a box containing q and having its sides normal and parallel to v . In the first frame, we get $\int E \cdot dA = q / \epsilon_0$. Now let's apply it in the second frame K' .

We have to be cautious here since the sides of the box parallel to v will shrink by the

factor $\sqrt{1 - \frac{v^2}{c^2}}$. To get the same result of integrating $\int E \cdot dA$ we have to assume that the

electric field in the direction normal v changes by the inverse of the same ratio to compensate the length contraction, while the component parallel to v is kept constant.

Hence: $E'_\perp = \gamma E_\perp, E'_{\parallel} = E_{\parallel}$.

What about the effect of the magnetic field in frame K on the electric field of K' ? Let's imagine a charge at rest in K' while moving in the static magnetic field of K , what will this charge feel? It will feel Lorentz force given by $F = q v/c \times B$. when a charge at rest feels a force that means that an electric field causing this force should exist in K' . Thus the total electric field in K' is given by :

$$\mathbf{E}'_\perp = \gamma (\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp), \quad E'_{\parallel} = E_{\parallel}$$

To write the magnetic field in K' in terms of fields in K we will use symmetry considerations. First we can write directly the electric field in K as

$E_{\perp} = \gamma(E_{\perp}' - \mathbf{V} \times \mathbf{B}_{\perp}')$, by substitution in the previous equation we get:

$E_{\perp}' = \gamma^2 E_{\perp} - \gamma^2 \mathbf{V} \times \mathbf{B}_{\perp}' + \gamma \mathbf{V} \times \mathbf{B}_{\perp}$. By solving for B' we get:

$B_{\perp} = \gamma(B_{\perp}' + \frac{\mathbf{V} \times \mathbf{E}_{\perp}'}{c^2})$. By using symmetry considerations again we can express B' as:

$B_{\perp}' = \gamma(B_{\perp} - \frac{\mathbf{V} \times \mathbf{E}_{\perp}}{c^2})$. Since the parallel component of the magnetic field results from

currents flowing normal to the direction of velocity which are the same in both frames, the parallel components are also the same. Hence $B_{\parallel}' = B_{\parallel}$. Note: these transformations

hold in SI units. For Gaussian units, c^2 will be replaced by c in the equation for B_{\perp}' while v will be replaced by v/c in the equation for E_{\perp}' .

From the electromagnetic field transformation we see that E, B don't transform like the spatial part of a four vector so we need a more complex representation to represent the EM field transformation in a form similar to the four vector transformation mentioned in the last section ; this is the tensor notation. The electromagnetic field tensor is a single entity that combines both the electric and magnetic field components. If we defined the electromagnetic field tensor ($F^{\mu\nu}$) in Gaussian units as

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

we can see that the matrix components transform as: $F'^{\mu\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} F^{\alpha\beta}$

where the elements Λ^{μ}_{ν} are the same elements of the Lorentz transformation of four vectors. Thus for a relative velocity in the x-direction we can see that the electromagnetic field tensor in frame K' expressed as :

$$\begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}$$

is equivalent to:

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And this yields immediately:

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - \beta B_z) & B'_y &= \gamma(B_y + \beta E_z) \\ E'_z &= \gamma(E_z + \beta B_y) & B'_z &= \gamma(B_z - \beta E_y) \end{aligned}$$

By defining the four current $J^{\mu} = (\rho, J_x, J_y, J_z)$ and the dual electromagnetic field tensor ($G^{\mu\nu}$) as

$$\begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix}$$

We can write the four Maxwell's equations in the following two simple equations:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0. \quad [\text{See Griffiths, p.504}].$$

Now we are in a position to check the invariance of Maxwell's equations under Lorentz transformation. Given Maxwell's equations in frame K as: $\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$, we want to prove that the

electromagnetic field tensors F', G' in frame K' obey the equations: $\frac{\partial F'^{\mu\nu'}}{\partial x'^{\nu'}} = \mu_0 J'^{\mu'}$,

$$\frac{\partial G'^{\mu\nu'}}{\partial x'^{\nu'}} = 0. \quad \text{Using the transformation } F'^{\mu\nu'} = \frac{\partial x^{\mu'}}{\partial x^\alpha} \frac{\partial x^{\nu'}}{\partial x^\beta} F^{\alpha\beta} \text{ we can write the LHS of the}$$

$$\text{first equation as } \frac{\partial F'^{\mu\nu'}}{\partial x'^{\nu'}} = \frac{\partial x^{\mu'}}{\partial x^\alpha} \frac{\partial x^{\nu'}}{\partial x^\beta} \frac{\partial F^{\alpha\beta}}{\partial x^{\nu'}} = \frac{\partial x^{\mu'}}{\partial x^\alpha} \frac{\partial F^{\alpha\beta}}{\partial x^\beta}.$$

And writing the Maxwell's equations in K as: $\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = \mu_0 J^\alpha$, and the transformation

$$\text{relation : } J'^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\alpha} J^\alpha, \text{ then we can write } \frac{\partial F'^{\mu\nu'}}{\partial x'^{\nu'}} = \mu_0 J'^{\mu'}$$

and hence the first Maxwell's equation is covariant under Lorentz transformation. By a similar way the second Maxwell's equation can be proved covariant.

Note: The constants ϵ_0, μ_0 are assumed to have the same values in all inertial frames since they are properties of free space.

Four-vector potential

Since we have written Maxwell's equation in a terms of the electromagnetic tensor let's try to define a four vector potential A^α that satisfies the covariant form: $\partial_\alpha A^\alpha = 0$. It's clear that this vector should consist of the electric potential Φ and the magnetic potential A. But we know that the magnetic potential can be defined upto an arbitrary gradient field while the electric potential can be defined upto an additive constant. That means that we have some freedom in choosing A and Φ . A Lorenz gauge is chosen as

$$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \quad \text{This implies that a four vector potential defined as } (\Phi, \mathbf{A}) \text{ satisfies}$$

$$\partial_\alpha A^\alpha = 0. \quad \text{By substituting by this gauge in Maxwell's equations we get the potential}$$

wave equations:

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi \rho$$

Which can be combined in terms of four vector quantities into the single equation:

$\nabla^2 A^\alpha = \frac{4\pi}{c} J^\alpha$, where ∇^2 is the four dimensional Laplacian operator

$$\nabla^2 = \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{02}} - \nabla^2.$$

With this definition of the four vector potential we can write the x-components of E,B as

$$E_x = -\frac{1}{c} \frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial t} = -(\partial^0 A^1 - \partial^1 A^0)$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$

Which can be generalized to write the electromagnetic field tensor in terms of the four-vector potential as: $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$.

2.2 Relativistic Electromagnetic Lagrangian

Lagrangian mechanics provides another mean to describe the kinematics of any system in a different way than Newtonian mechanics. An action for any system is defined as the integral of the lagrangian functional of the path the system may take in the space of solutions.

$$A[\mathbf{q}] = \int_{t_i}^{t_f} L(\mathbf{q}, \dot{\mathbf{q}}; t) dt,$$

The solution (or the path the system will take) is the one that makes the value of the action an extreme. (i.e, $0 = \delta A[\mathbf{q}]$). This leads to the Euler's-Lagrange equations which fully describe the motion of the system

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^j} \right) = \frac{\partial L}{\partial q^j},$$

The equation that describe the motion of a charged particle interacting with an

electromagnetic field is: $\frac{dp}{dt} = e \left[E + \frac{u}{c} \times B \right]$. Any Lagrangian formulation should yield

an equivalent equation of motion. So, let's seek the proper Lagrangian that satisfies this requirement in addition to the relativistic requirements. The principle of relativity requires that the action should be an extreme in all inertial frames. If the action has a value in one frame lower than in another frame, that means our definition of the Lagrangian in the second frame didn't yield a minimum value, and the system could have acquired the relative velocity between the two frames to achieve a lower action and that violates the principle of least action in the second frame. Hence the Lagrangian should be the same in all inertial frames. Let's define the action in terms of the proper

time (the time in the inertial frame of the system) as $A = \int_{\tau_1}^{\tau_2} \gamma \mathcal{L} d\tau$ hence the quantity $\gamma \mathcal{L}$

should be invariant in all inertial frames. (In a frame different than the inertial frame the Euler equations will be written in terms of L not $\gamma \mathcal{L}$). For a free particle in a

nonrelativistic proplem we know that the Lagrangian is the kinetic energy of the

particles which equals : $L = \frac{1}{2} m v^2$ when the equation of motion is written we obtain

Newton's first law, namely: $m \ddot{\vec{v}} = \mathbf{0}$. Now a relativistic Lagrangian should yield the conservation law of momentum in the relativistic form($\mathbf{P} = \gamma m \mathbf{v}$). Where m is the rest mass of the particle. Since the quantity $\gamma \mathcal{L}$ is lorentz invariant we can see that L

takes the form $\alpha \gamma^{-1}$ where α is constant. The logical choice of α is $-mc^2$ where m is the rest mass of the particle. By writing the Euler equation for this L we get the

correct form for conservation of momentum. $\frac{d}{dt}(\gamma mu) = 0$. Thus $L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}$.

Now we have to add to the lagrangian a term that describes the interaction energy between a single charge and the electromagnetic field. We know that the classical interaction energy for a distribution of charges and currents is expressed as:

$$W_{\text{int}} = \int_V [\rho(\mathbf{r}, t)\Phi_{\text{ext}}(\mathbf{r}, t) - \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{A}_{\text{ext}}(\mathbf{r}, t)] d^3x.$$

We want to add a term analogous to this in the Lagrangian which when multiplied by γ becomes an invariant quantity. (i.e L_{int} should be γ^{-1} multiplied by a lorentz invariant quantity. Returning to

Gaussian units, a logical choice is: $L_{\text{int}} = -\frac{e}{\gamma c} U_\alpha A^\alpha$ where U is the four vector velocity

$(\gamma c, \gamma \mathbf{u})$. This yields the expected interaction lagrangian : $L_{\text{int}} = -e\Phi + \frac{e}{c}(\mathbf{u} \cdot \mathbf{A})$. So the total lagrangian now (excluding the energy of the fields) becomes

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \vec{A} \cdot \vec{v} - e\phi.$$

By applying Euler's equations on this Lagrangian, the equation of motion for the particle $\frac{d(m_0 \gamma \mathbf{v})}{dt} = e \left[\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right]$ will be obtained. We

note here that this Lgrangian doesn't consider the finite speed of propagation of electromagnetic potentials. By writing the Hamiltonian of the charged particle, defined by $H = \mathbf{P} \cdot \mathbf{u} - L$ and by defining the conjugate momentum $P_i = \frac{\partial L}{\partial u_i} = \gamma m u_i + \frac{e}{c} A_i$ we find

that the Hamiltonian of the particle is: $H = \sqrt{(cP - eA)^2 + m^2 c^4} + e\Phi$. By letting the

total energy be W and squaring both sides, we get $(W - e\Phi)^2 - (cP - eA)^2 = (mc^2)^2$ which can be rewritten in terms of the four momentum vectors as $P_\alpha P^\alpha = (mc)^2$ where

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p} \right) = \left(\frac{1}{c}(W - e\Phi), P - \frac{e}{c} \mathbf{A} \right).$$

This is called the minimal substitution. Now that we derived the Lagrangian for a particle interacting with an electromagnetic field let's try to find the lagrangian for the field component themselves ($F_{\beta\alpha}$) which when used in Euler-Lagrange equations leads to the equations of motion to the field tensor found before (Maxwell's equations). First we note that when we treat continuous field systems by the Lagrangian approach, the finite number of coordinates $q_i(t)$ are replaced by an infinite number of degrees of freedom. For each point in space time X^α a limited number of field components $\phi_K(x)$ are assigned. This field changes continuously

through space and time. Thus Euler equations takes the form: $\partial^\beta \frac{\partial L}{\partial(\partial^\beta \phi_K)} = \frac{\partial L}{\partial \phi_K}$. The

lagrangian again should be a Lorentz invariant quantity (i.e, Lorentz scalar). A proposed one is.

$$\mathcal{L} = -j^\mu A_\mu - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}$$

To be able to apply Euler equation, we will write the field tensor in terms of the magnetic potential $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, hence the Lagrangian becomes:

$$-j^\mu A_\mu - \frac{1}{4\mu_0} (\partial^\mu A^\nu - \partial^\nu A^\mu) g_{\mu\mu'} g_{\nu\nu'} (\partial^{\mu'} A^{\nu'} - \partial^{\nu'} A^{\mu'}).$$

And Euler equations takes the form:

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu A^\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0.$$

Solving this equation leads to the Maxwell's equations:

$$-\frac{1}{\mu_0} \partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) - (-j_\nu) = 0$$

Putting it in in the tensor form we get $\partial^\mu F_{\mu\nu} = \mu_0 j_\nu$.

which is the same equation shown in the previous section.

If we expanded the four vector potential and current and the EM tensor in the Lagrangian we can write the Lagrangian as

$$\begin{aligned} &= -j^\mu A_\mu + \frac{1}{4\mu_0} \text{Tr} \left(\begin{pmatrix} 0 & -\frac{1}{c} E_x & -\frac{1}{c} E_y & -\frac{1}{c} E_z \\ \frac{1}{c} E_x & 0 & -B_z & B_y \\ \frac{1}{c} E_y & B_z & 0 & -B_x \\ \frac{1}{c} E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{c} E_x & \frac{1}{c} E_y & \frac{1}{c} E_z \\ -\frac{1}{c} E_x & 0 & -B_z & B_y \\ -\frac{1}{c} E_y & B_z & 0 & -B_x \\ -\frac{1}{c} E_z & -B_y & B_x & 0 \end{pmatrix} \right) \\ &= -j^\mu A_\mu + \frac{2}{4\mu_0} \left(\frac{1}{c^2} E^2 - B^2 \right) = -\rho \Phi + \vec{j} \cdot \vec{A} + \frac{1}{2} \epsilon_0 E^2 - \frac{1}{2\mu_0} B^2. \end{aligned}$$

That means that $E^2 - c^2 B^2$ is an invariant quantity.

Now let's try to apply some of the concepts learned so far to some simple problems.

Consider two electrons moving parallel to each other with the same velocity v w.r.t a stationary observer. The interaction between the 2 electrons should yield the same kind of motion subjected to time dilation effects when analyzed from both frames the stationary frame and the moving frame. In the frame of the 2 electrons each one of them

is subjected to a Coulomb repulsive force of magnitude $\frac{e^2}{4\pi\epsilon r^2}$. Thus each electron will

has an acceleration in the y-axis of magnitude F/m and its y-component will be of the form $a^*t^2/2$. In the second frame both electrons are subject to a Lorentz force

$F = e(E_\perp + v \times B_\perp)$. Using the transformation relations developed $E'_\perp = \gamma E_\perp$,

$B'_\perp = \gamma(B_\perp + \frac{v \times E_\perp}{c^2})$ where $B_\perp = 0$, $E_\perp = \frac{e}{4\pi\epsilon r^2}$. (Note we used a plus sign since we

are transforming from the moving frame to the rest frame) By substituting in the

expression of F we find $F = eE'_\perp \gamma (1 - \frac{v^2}{c^2}) = eE'_\perp \gamma^{-1} = F' \gamma^{-1}$. Now if we tried to analyze

the motion of the charge in the normal direction using the Newton equation $F=ma$ and taking care of the relativistic transformation of mass $m' = m\gamma$ we find

$F' = \gamma^2 m a' = \gamma^2 m \frac{d^2 y}{dt'^2}$. Where F' is the force in the moving frame. And since we know

that the time interval in the moving frame dt' is related to the time interval in the stationary frame dt by $dt' = \gamma dt$ we obtain the same equation of motion in the first frame by transforming the time in the second derivative. Thus the equation of motion is the same in both frames subjected to the time dilation phenomenon. We conclude from this example that a pure electrostatic phenomenon in one frame may be a combination of both electric and magnetic in another frame. Notice that a nonrelativistic treatment of

this problem would yield two different forces in both frames and hence two different equations of motion. Some authors even interpret the force between two current carrying conductors as a relativistic effect due to the movement of the charges inside the conductor. (e.g, Purcell in his “Electricity and Magnetism”).

Conclusion

The field of classical electrodynamics still has a lot to be done for it. In the most general case, two charges interacting with each other are accelerated. Neither of the charges exists in an inertial frame, and both charges will radiate according to Maxwell’s equations. One deficiency of classical electrodynamics in its current formalism is that we need two fields (electric and magnetic that are calculated separately) to describe the force acting on a moving charge. I think we should seek a single force relating the interaction between two charges in the most general situation in terms of their positions and relative velocities. The fact that the electric field emanating from a uniformly moving charge is always radial and emanates from the current position of the charge (not a retarded position) is somewhat surprising. It implies that that several pieces of information are transmitted along with the electromagnetic field (at the speed of light). I hope that we could reach a more general formulation of the interaction between two charges that is valid even in the non-inertial frames of the charges.

The electric force and magnetic force seem to have separate physical origins in current electrodynamics. One results from the mere existence of charges and another results from the motion of charges. I think a complete unification of electricity and magnetism should interpret both electric and magnetic forces from a single physical origin. Though the force of gravity is similar to Coulomb’s force in form, we don’t see the effect of magnetic force between moving masses as we have it between moving charges. A general relativistic prediction called frame dragging predicts such type of interaction.

References

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