

Term 081  
KFUPM  
EE 656: Robotics & Control

# *Homework 4*

*Spherical Pendulum with Spring Simulation*

By

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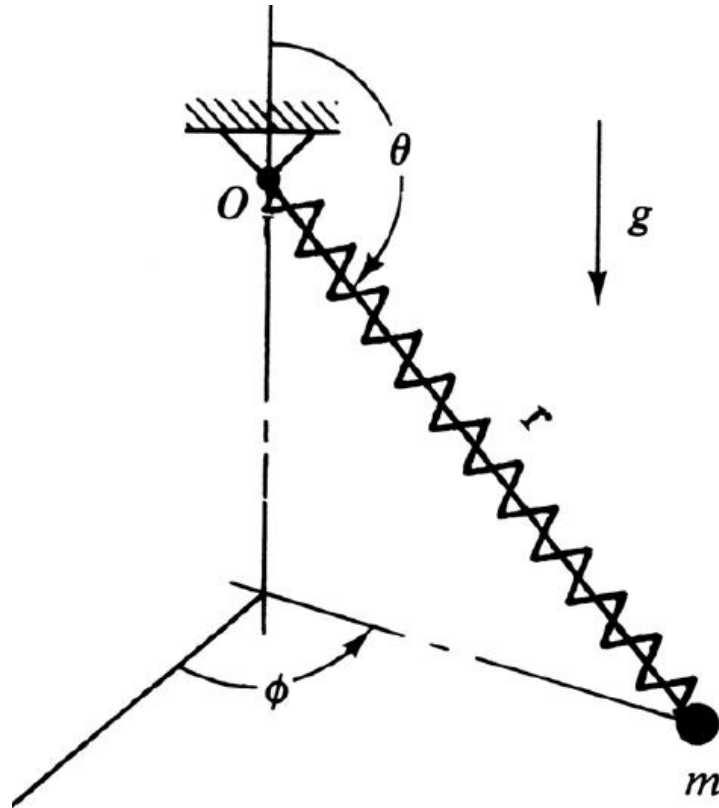
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For

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HW 4



### 1. Dynamic Equations

The above Spherical Pendulum with spring has its fixed point at the origin  $(0, 0, 0)$ . With configuration variables of

$$q = \begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}$$

As described in figure above.

To form the equations of motion, we utilize Lagrange Dynamics.

**Kinetic Energy:**

$$KE = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 + \frac{1}{2}mv_\phi^2$$

With

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$v_\phi = r \sin(\pi - \theta) \dot{\phi}$$

So,

$$\Rightarrow KE = \frac{m}{2} [\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2(\pi - \theta) \dot{\phi}^2]$$

**Potential Energy:**

$$PE = \frac{1}{2}K(r - L)^2 - mgr \cos(\pi - \theta)$$

To have the Lagrangian:

$$\mathcal{L} = KE - PE$$

$$= \frac{m}{2} [\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2(\pi - \theta) \dot{\phi}^2] - \frac{1}{2}K(r - L)^2 + mgr \cos(\pi - \theta)$$

As

$$\sin(\pi - \theta) = \sin \theta \text{ \&}$$

$$\cos(\pi - \theta) = -\cos \theta$$

$$\Rightarrow \mathcal{L} = \frac{m}{2} [\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2(\theta) \dot{\phi}^2] - \frac{1}{2}K(r - L)^2 - mgr \cos(\theta)$$

As generally,

$$f_q = \frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}} \right] - \frac{\partial \mathcal{L}}{\partial q}$$

So, equations of motion of the system becomes

$$f_r = \frac{d}{dt}[m\dot{r}] - [m\dot{\theta}^2 r + m \sin^2(\theta) \dot{\phi}^2 r - K(r - L) - mg \cos(\theta)]$$

$$\Rightarrow f_r = m\ddot{r} - m\dot{\theta}^2 r - m \sin^2(\theta) \dot{\phi}^2 r + K(r - L) + mg \cos(\theta)$$

$$f_\theta = \frac{d}{dt}[mr^2\dot{\theta}] - [mr^2\dot{\phi}^2 \sin(\theta) \cos(\theta) + mgr \sin(\theta)]$$

$$\Rightarrow f_\theta = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\dot{\phi}^2 \sin(\theta) \cos(\theta) - mgr \sin(\theta)$$

$$f_\phi = \frac{d}{dt}[mr^2 \sin^2(\theta) \dot{\phi}]$$

$$\Rightarrow f_\phi = m \left[ r^2 \sin^2(\theta) \ddot{\phi} + \dot{\phi} \frac{d}{dt}[r^2 \sin^2(\theta)] \right]$$

$$\Rightarrow f_\phi = mr^2 \sin^2(\theta) \ddot{\phi} + 2mr^2 \sin(\theta) \cos(\theta) \dot{\theta} \dot{\phi} + 2mr\dot{r} \sin^2(\theta) \dot{\phi}$$

So the complete Equations of Motion are:

$$\begin{bmatrix} m\ddot{r} - m\dot{\theta}^2 r - m \sin^2(\theta) \dot{\phi}^2 r + K(r - L) + mg \cos(\theta) \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\dot{\phi}^2 \sin(\theta) \cos(\theta) - mgr \sin(\theta) \\ mr^2 \sin^2(\theta) \ddot{\phi} + 2mr^2 \sin(\theta) \cos(\theta) \dot{\theta} \dot{\phi} + 2mr\dot{r} \sin^2(\theta) \dot{\phi} \end{bmatrix} = \begin{bmatrix} f_r \\ f_\theta \\ f_\phi \end{bmatrix}$$

The above equation can be put in the form of

$$\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2 r + \sin^2(\theta) \dot{\phi}^2 r - \frac{K}{m}(r - L) - g \cos(\theta) + \frac{f_r}{m} \\ -\frac{2\dot{r}\dot{\theta}}{r} + \dot{\phi}^2 \sin(\theta) \cos(\theta) + \frac{g \sin(\theta)}{r} + \frac{f_\theta}{mr^2} \\ -2 \cot(\theta) \dot{\theta} \dot{\phi} - \frac{2\dot{r}\dot{\phi}}{r} + \frac{f_\phi}{mr^2 \sin^2(\theta)} \end{bmatrix}$$

Form a states equation with states

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \\ \varphi \\ \dot{\varphi} \end{bmatrix}$$

Then, the equation of motion becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ -\frac{2x_2x_4}{x_1} + x_6^2 \sin(x_3) \cos(x_3) + \frac{g \sin(x_3)}{x_1} \\ x_6 \\ 2 \cot(x_3) x_4 x_6 - \frac{2x_2x_6}{x_1} \\ x_2 \\ x_4 \\ -\frac{K}{m}(x_1 - L) - g \cos(x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f_r}{m} \\ 0 \\ \frac{f_\theta}{mx_1^2} \\ 0 \\ \frac{f_\varphi}{mx_1^2 \sin^2(x_3)} \end{bmatrix}$$

The above equation describes fully the motion of the Spherical Pendulum with spring system.

Let us also form an output vector to give us coordinates in XYZ space:

$$y = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r \sin(\theta) \cos(\varphi) \\ r \sin(\theta) \sin(\varphi) \\ r \cos(\theta) \end{bmatrix} = \begin{bmatrix} x_1 \sin(x_3) \cos(x_5) \\ x_1 \sin(x_3) \sin(x_5) \\ x_1 \cos(x_3) \end{bmatrix}$$

## 2. System Simulation

Having the above system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4^2 x_1 + \sin^2(x_3) x_6^2 x_1 - \frac{K}{m}(x_1 - L) - g \cos(x_3) \\ x_4 \\ -\frac{2x_2 x_4}{x_1} + x_6^2 \sin(x_3) \cos(x_3) + \frac{g \sin(x_3)}{x_1} \\ x_6 \\ 2 \cot(x_3) x_4 x_6 - \frac{2x_2 x_6}{x_1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f_r}{m} \\ 0 \\ \frac{f_\theta}{mx_1^2} \\ 0 \\ \frac{f_\varphi}{mx_1^2 \sin^2(x_3)} \end{bmatrix}$$

$$y = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x_1 \sin(x_3) \cos(x_5) \\ x_1 \sin(x_3) \sin(x_5) \\ x_1 \cos(x_3) \end{bmatrix}$$

With values of parameters of

$$L = m = K = 1$$

$$g = 9.8$$

With initial position of

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This corresponds to states initial values of (assuming free fall from rest),

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \\ x_6(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \pi/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Also, the input forces into the system is put as

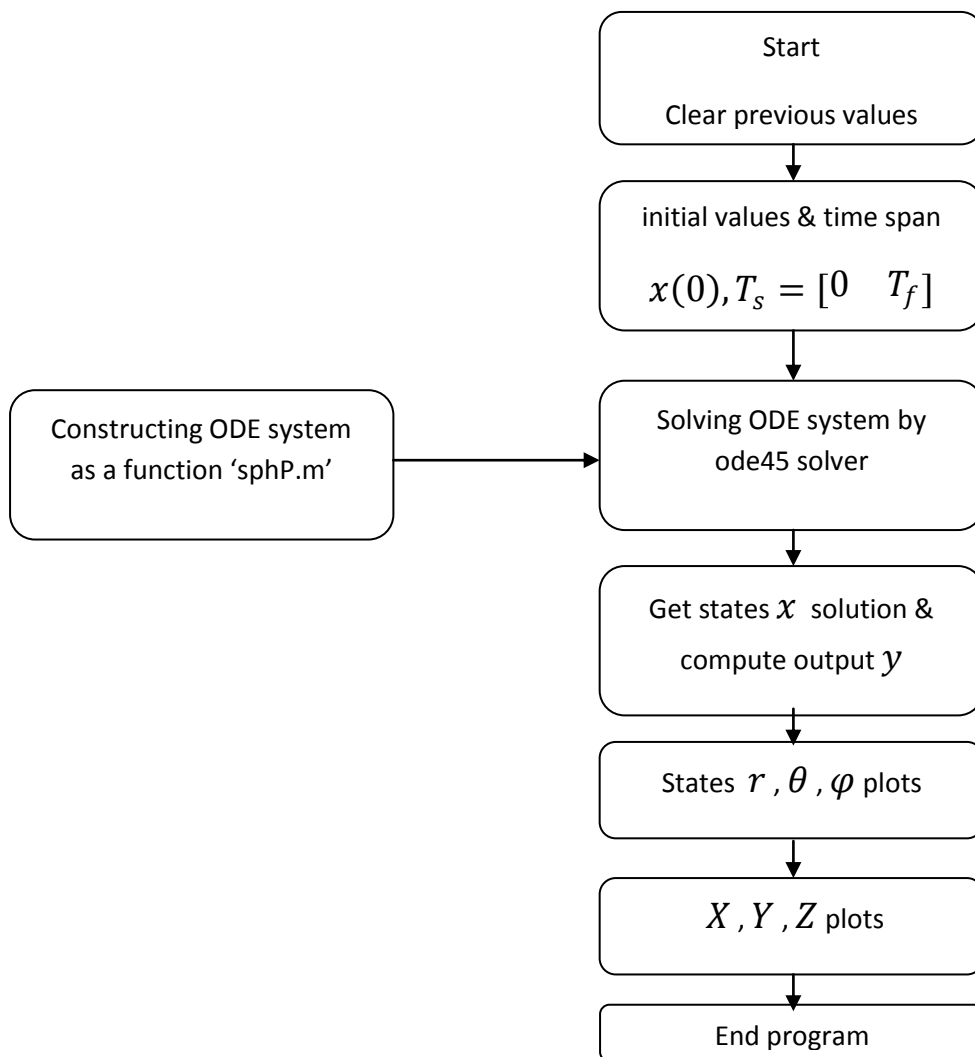
$$f_r = -\dot{r} = -x_2$$

$$f_\theta = -\dot{\theta} = -x_4$$

$$f_\varphi = -\dot{\varphi} = -x_6$$

### Simulation Program:

Using MATLAB R2007b, the simulation utilizes the “ordinary Differential Equations” solvers in MATLAB. The flow chart of the program goes as follows,



Complete program is found in appendix.

## Simulation Results:

Here are the plots of the configuration variables of the systems:

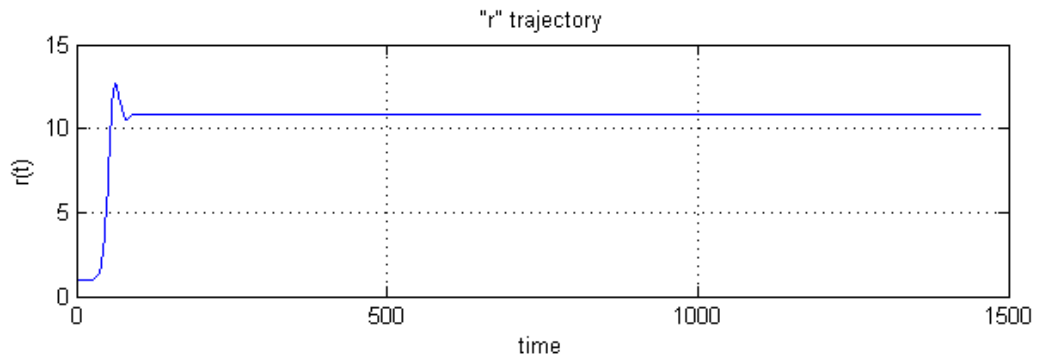


Figure 1- trajectory of 'R'

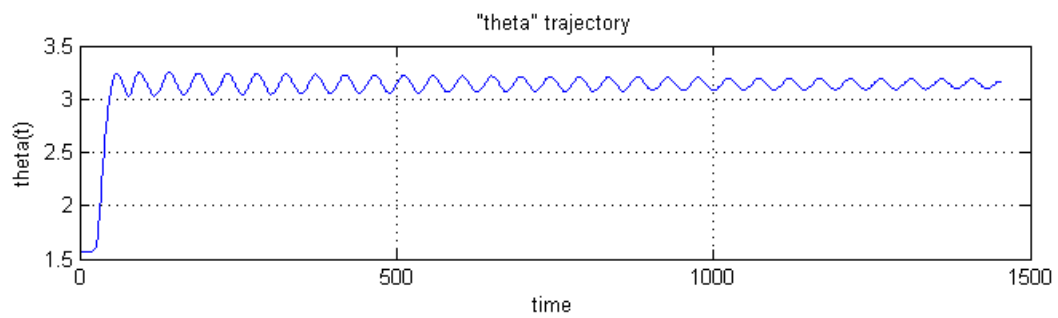


Figure 2- trajectory of 'theta'

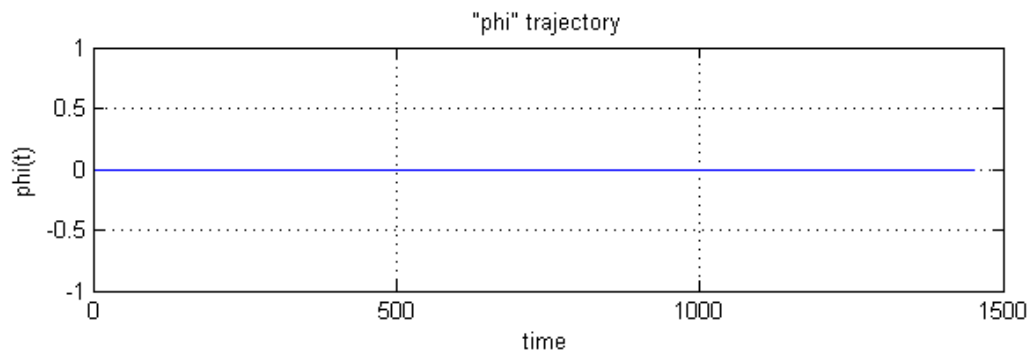


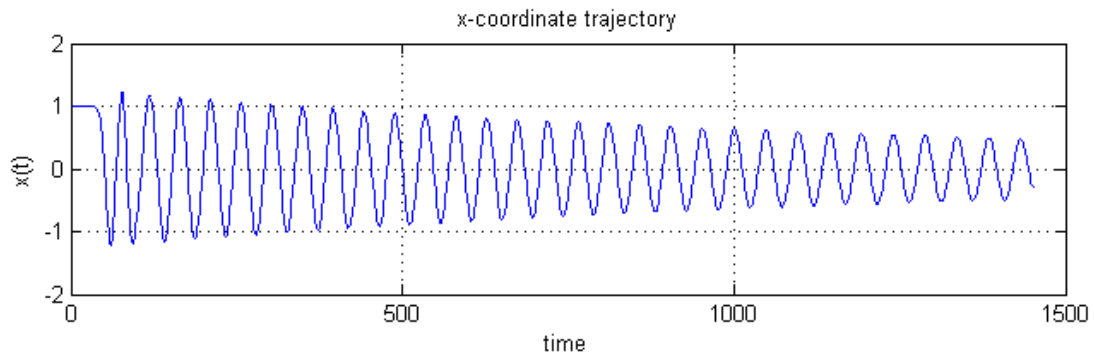
Figure 3- trajectory of 'phi'

From above plots, we can see that

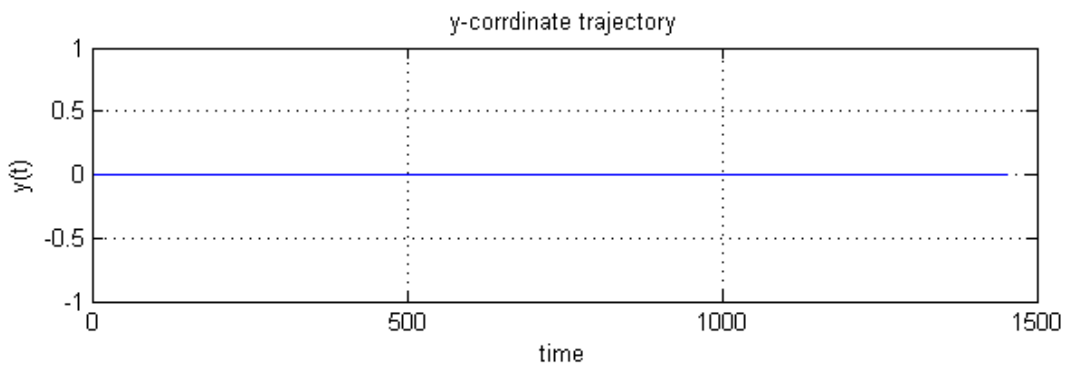
- Due to the initial conditions, we have no motion along the  $\varphi$  rotation axis.



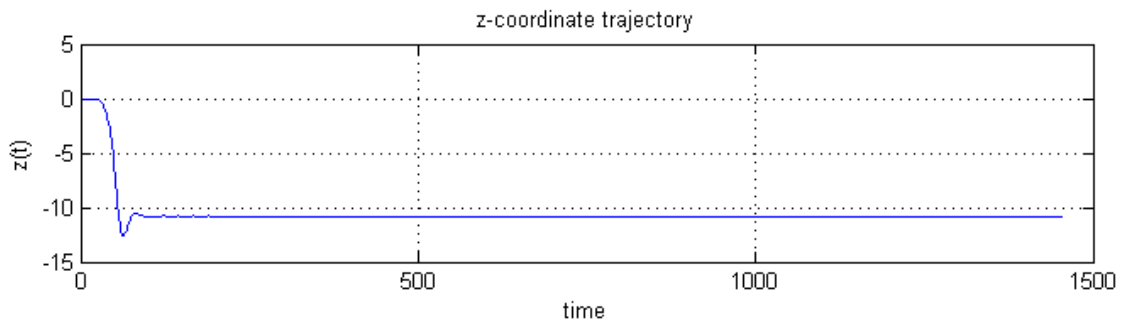
Below, you can see the plots of the XYZ space variables



**Figure 4- X-coordinate**



**Figure 5- Y-coordinates**



**Figure 6- Z-coordinate**

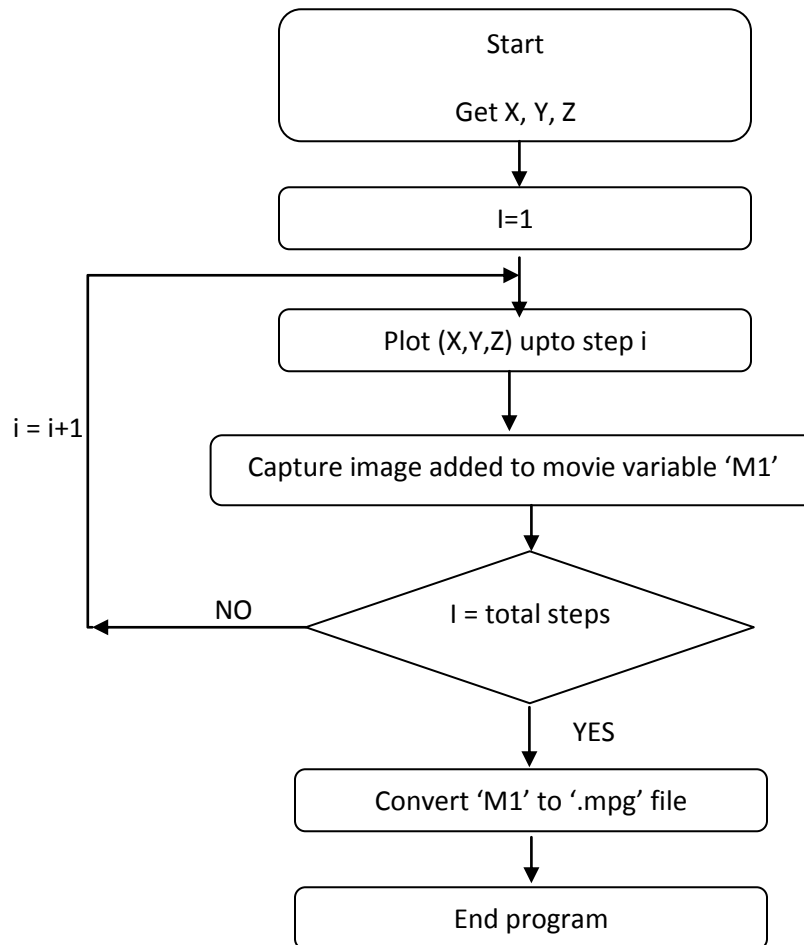
From above plots, we can see that,

- as there is no interaction between  $\varphi, \theta$  (no coupling), the Y-position has no change throughout the motion

### 3. Motion Animation

Here in this section, we describe the program of generating the animation of motion for the Spherical Pendulum with spring system. This animation program assumes receiving the computed trajectories of variables from previous step,

The flow chart of the program,



#### Animation Output:

2 animations are generated:

- 1) 3D animation of the pendulum
- 2) Side-view animation (Z vs. X)

Below, you can find the overall trajectory in space for both 3D- & XZ- space.

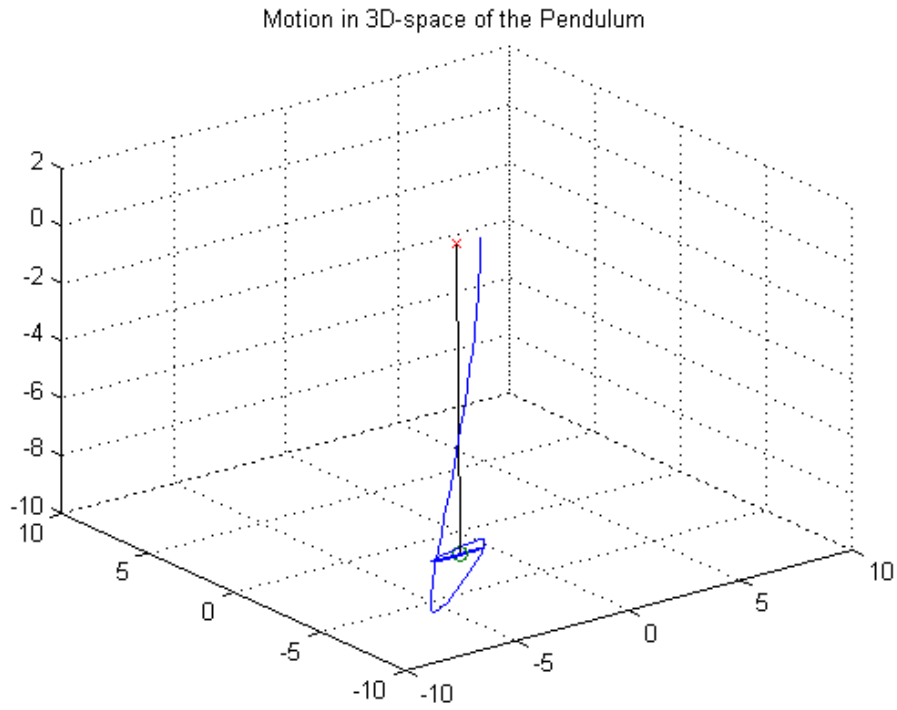


Figure 7- Trajectory in 3D space

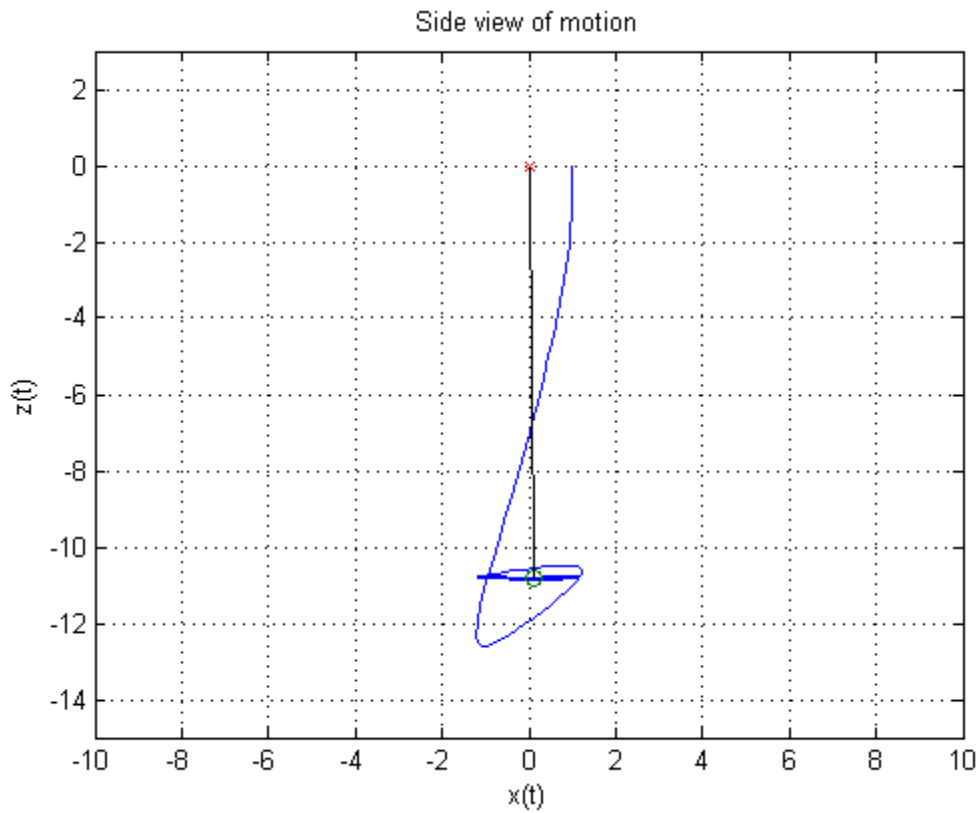


Figure 8- Z vs. X overall trajectory

In above plots:

- A small circle corresponds to the point mass of the pendulum
- The **black** straight line corresponds to the pendulum link
- The **blue** curve corresponds to the trajectory of the pendulum

### **Animation files:**

Two '.mpg' files are generated for the complete motion of the pendulum.

Complete program is found in Appendix

## Appendix

### 1) 'sphP.m': function that contain the System Equation of Motion

```
function xdot=sphP(t,x)

xdot=zeros(6,1);

K=1;
m=1;
L=1;

g=9.8;

fx=-x(2);
xdot(1)=x(2);
xdot(2)=(x(4)^2)*x(1)+(sin(x(3))^2)*(x(6)^2)*x(1)-(K/m)*(x(1)-L)-
g*cos(x(3))+fx/m;

fth=-x(4);
xdot(3)=x(4);
xdot(4)=-
2*x(2)*x(4)/(x(1))+(x(6)^2)*sin(x(3))*cos(x(3))+g*sin(x(3))/x(1)+fth/(m
*x(1)^2);

fphi=-x(6);
xdot(5)=x(6);
xdot(6)=-2*cot(x(3))*x(4)*x(6)-
2*x(2)*x(6)/x(1)+fphi/(m*(x(1)^2)*sin(x(3))^2);
```

### 2) Program of System Simulation

```
close all
clear all
clc
%% Initalization
x0=[1 0 pi/2 0 0 0]; %states initial values
Ts=[0 200]; %time span

%% ODE solving
[T,Y] = ode45(@sphP,Ts,x0);

%% output
y1=Y(:,1).*sin(Y(:,3)).*cos(Y(:,5)); % X
y2=Y(:,1).*sin(Y(:,3)).*sin(Y(:,5)); % Y
y3=Y(:,1).*cos(Y(:,3)); % Z

%% Plots
figure
plot(Y(:,1))
grid
title('"r" trajectory')
```

```

xlabel('time')
ylabel('r(t)')

figure
plot(Y(:,3))
grid
title('"theta" trajectory')
xlabel('time')
ylabel('theta(t)')

figure
plot(Y(:,5))
grid
title('"phi" trajectory')
xlabel('time')
ylabel('phi(t)')

%%% output plots
figure
plot(y1)
grid
title('x-coordinate trajectory')
xlabel('time')
ylabel('x(t)')

figure
plot(y2)
grid
title('y-corrdinate trajectory')
xlabel('time')
ylabel('y(t)')

figure
plot(y3)
grid
title('z-coordinate trajectory')
xlabel('time')
ylabel('z(t)')

```

### 3) Motion Animation program

```

%% setting frames speed
d=7;
j=1:d:length(T);

%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% trajectories of y1, y2 , y3 must available in WORKSPACE
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% generating images in 3D
figure
for i=1:length(j)-1
    hold off
    %plot in 3D

```

```

plot3(y1(1:j(i)),y2(1:j(i)),y3(1:j(i)),y1(j(i)),y2(j(i)),y3(j(i)),'o',[
0 y1(j(i))],[0 y2(j(i))],[0 y3(j(i))],'k',0,0,0,'xr')
    title('Motion in 3D-space of the Pendulum')
    axis([-10 10 -10 10 -10 2]);
    grid
    hold on
    M1(i)=getframe(gcf);
end
drawnow;

%% exporting to 'mpg' movie
mpgwrite(M1,'RGB','pendulum3D.mpg')

%% generating images in XZ
figure
for i=1:length(j)-1
    hold off
    plot(y1(1:j(i)),y3(1:j(i)),y1(j(i)),y3(j(i)),'o',[0 y1(j(i))],[0
y3(j(i))],'k',0,0,'xr')
    title('Side view of motion')
    xlabel('x(t)')
    ylabel('z(t)')
    axis([-10 10 -15 3]);
    grid
    hold on
    M1(i)=getframe(gcf);
end
drawnow;

%% expoting to 'mpg'
mpgwrite(M1,'RGB','pendulumXZ.mpg')

```